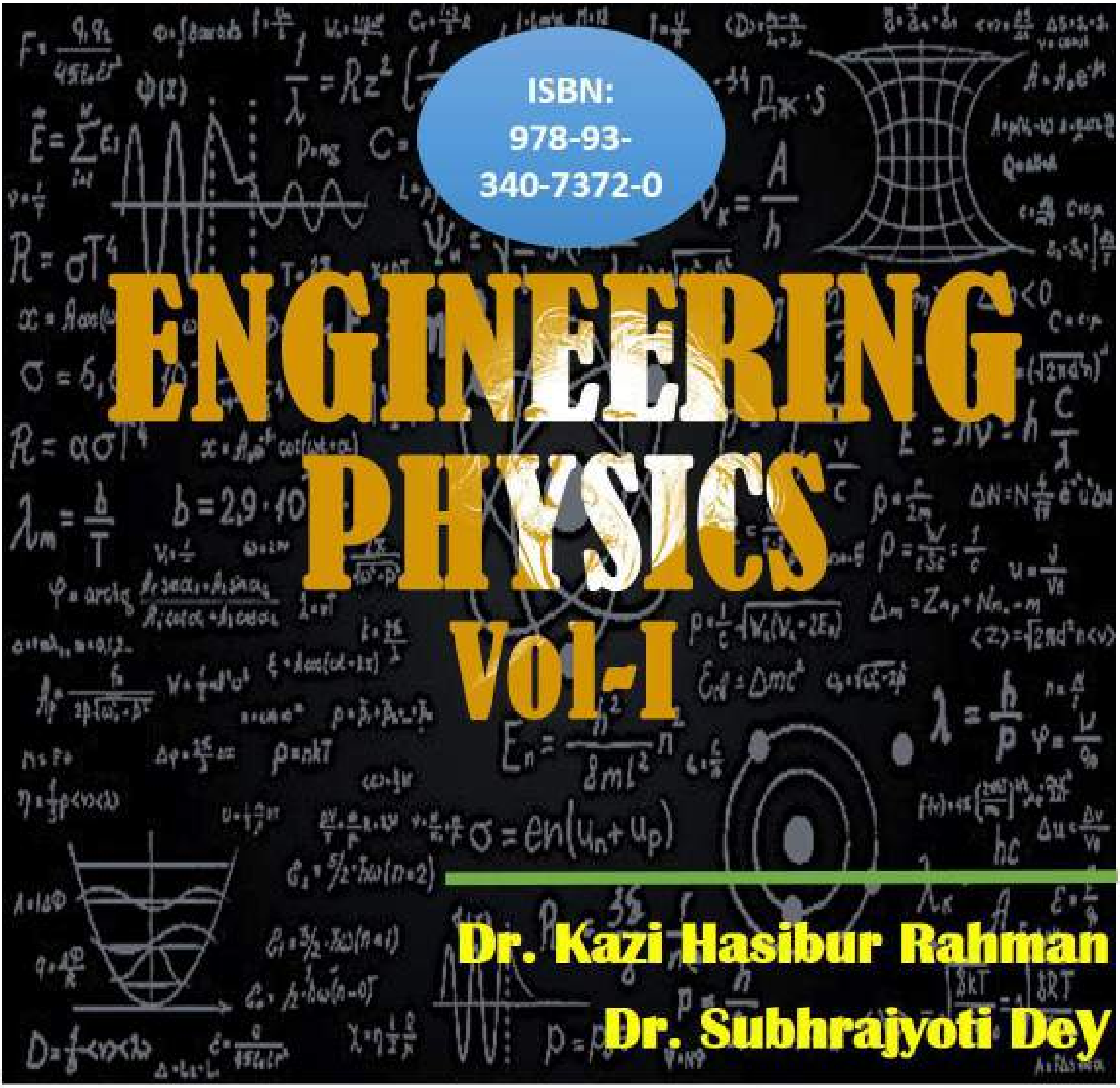


ISBN:
978-93-
340-7372-0

ENGINEERING PHYSICS

Vol-I

Dr. Kazi Hasibur Rahman

Dr. Subhrajyoti Dey

Swami Vivekananda University, Kolkata

Swami Vivekananda University, Kolkata (Institutional Publisher)

Published by the Swami Vivekananda University (Institutional Publisher), Kolkata-700121,

West Bengal, India

Swami Vivekananda University (Institutional Publisher), Kolkata-700121, India.

ISBN: 978-93-5967-750-7

First Edition: July, 2024

Publisher: Swami Vivekananda University (Institutional Publisher), Kolkata-700121, India

Contact us: hasiburr@svu.ac.in

Preface

The purpose of this book is to provide an in-depth information on fundamentals of Engineering Physics to the student community to improve their general understanding on the subject and designed as a textbook for the beginners in all branches of Engineering according to the recent syllabus of Swami Vivekananda University, Kolkata. The book has been divided into four Chapters.

The first Chapter is Electromagnetism, which contains Maxwell's Equation, Electrostatics, Electrostatics potential, Field lines, Electrostatics energy, Electric Dipoles, Davisson and Germer Experiment, Matter waves and their Characteristic properties, and Expression for de-Broglie wavelength using group velocity. **The second Chapter** is on Dielectric & Magnetic Properties of Materials, which includes Dielectric constant and polarization of dielectric materials. Types of polarization. Equation for internal fields in liquids and solids, Classius – Mussoti equation, Ferro and Piezo – electricity, Frequency dependence of dielectric constant, and Soft and Hard magnetic materials. **The third Chapter** is Quantum Mechanics, which involves Heisenberg's uncertainty principle and its physical significance, Wave function, Time independent Schrödinger wave equation, Application of Schrödinger wave equation – Energy eigen values for a free particle, and Energy eigen values of a particle in a potential well of infinite depth. **The fourth Chapter** is oscillations and vibrations, which involves oscillatory motion, Harmonic and non-harmonic function, SHM and their examples, Damped and forced vibration, Amplitude and velocity resonance, sharpness of resonance, steady state forced vibrations and rate of dissipation of energy due to resistive frictional force.

Every attempt has been made to make this book error free and useful for the students. Each Chapter begins with objective and ends with unit questions, assignment problems.

Any constructive suggestion and criticism regarding the improvement of this book will be acknowledged.

(Dr. Kazi Hasibur Rahman) Date: 03-07-2024
Assistant Professor,
Swami Vivekananda University, Kolkata, West Bengal, India

Acknowledgement

I would like to express my heartfelt appreciation to Swami Vivekananda University in Kolkata, India, for their invaluable support and encouragement in the creation of this book, **"Engineering Physics Vol-I"** The university's unwavering dedication to fostering research and excellence in teaching has been instrumental in shaping the focus and content of this publication. The collaborative environment and extensive resources provided by Swami Vivekananda University have enabled us to thoroughly explore and present the latest developments in engineering mathematics. This supportive setting has been crucial for conducting the necessary research and ensuring that the material presented in this textbook is both current and relevant to the needs of today's engineering students. I am particularly grateful for the university's commitment to academic excellence and innovation. Their support has allowed us to create a comprehensive and accessible resource that we hope will be beneficial to both students and educators in the field of engineering. It is my sincere hope that this book, reflecting our shared commitment to knowledge, progress, and the pursuit of excellence, will serve as a valuable tool for Swami Vivekananda University as well as the broader academic community. We look forward to continuing our partnership and contributing to the advancement of education and research in engineering mathematics. Thank you to everyone at Swami Vivekananda University who has contributed to this endeavour. Your support and encouragement have been greatly appreciated. Sincerely

(Dr. Kazi Hasibur Rahman)
Assistant Professor,
Swami Vivekananda University, Kolkata, West Bengal, India

CONTENTS

CHAPTER 1:

ELECTROMAGNETISM

Chapter 1

1. 1. Introduction

There are, to the best of our knowledge, four forces at play in the Universe. At the very largest scales — those of planets or stars or galaxies — the force of gravity dominates. At the very smallest distances, the two nuclear forces hold sway. For everything in between, it is force of electromagnetism that rules.

At the atomic scale, electromagnetism (admittedly in conjunction with some basic quantum effects) governs the interactions between atoms and molecules. It is the force that underlies the periodic table of elements, giving rise to all of chemistry and, through this, much of biology. It is the force which binds atoms together into solids and liquids. And it is the force which is responsible for the incredible range of properties that different materials exhibit.

At the macroscopic scale, electromagnetism manifests itself in the familiar phenomena that give the force its name. In the case of electricity, this means everything from rubbing a balloon on your head and sticking it on the wall, through to the fact that you can plug any appliance into the wall and be pretty confident that it will work. For magnetism, this means everything from the shopping list stuck to your fridge door, through to trains in Japan which levitate above the rail. Harnessing these powers through the invention of the electric dynamo and motor has transformed the planet and our lives on it.

As if this wasn’t enough, there is much more to the force of electromagnetism for it is, quite literally, responsible for everything you’ve ever seen. It is the force that gives rise to light itself.

Rather remarkably, a full description of the force of electromagnetism is contained in four simple and elegant equations. These are known as the Maxwell equations. There are few places in physics, or indeed in any other subject, where such a richly diverse set of phenomena flows from so little. The purpose of this course is to introduce the Maxwell equations and to extract some of the many stories they contain. However, there is also a second theme that runs through this course. The force of electromagnetism turns out to be a blueprint for all the other forces. There are various mathematical symmetries and structures lurking within the Maxwell equations, structures which Nature then repeats in other contexts. Understanding the mathematical beauty of the equations will allow us to see some of the principles that underly the laws of physics, laying the groundwork for future study of the other forces.

1.1.1 Charge and Current

Each particle in the Universe carries with it a number of properties. These determine how the particle interacts with each of the four forces. For the force of gravity, this property is mass. For the force of electromagnetism, the property is called electric charge.

For the purposes of this course, we can think of electric charge as a real number, $q \in R$. Importantly, charge can be positive or negative. It can also be zero, in which case the particle is unaffected by the force of electromagnetism.

The SI unit of charge is the Coulomb, denoted by C. It is, like all SI units, a parochial measure, convenient for human activity rather than informed by the underlying laws of the physics. (We'll learn more about how the Coulomb is defined in Section 3.5). At a fundamental level, Nature provides us with a better unit of charge. This follows from the fact that charge is quantised: the charge of any particle is an integer multiple of the charge carried by the electron which we denoted as −e, with

$$e = 1.60217657 \times 10^{-19} \text{ C}$$

A much more natural unit would be to simply count charge as $q = ne$ with $n \in Z$. Then electrons have charge −1 while protons have charge +1 and neutrons have charge 0. Nonetheless, in this course, we will bow to convention and stick with SI units. (An aside: the charge of quarks is actually $q = -e/3$ and $q = 2e/3$. This doesn't change the spirit of the above discussion since we could just change the basic unit. But, apart from in extreme circumstances, quarks are confined inside protons and neutrons so we rarely have to worry about this).

One of the key goals of this course is to move beyond the dynamics of point particles and onto the dynamics of continuous objects known as fields. To aid in this, it's useful to consider the charge density,

$$\rho(x, t)$$

defined as charge per unit volume. The total charge Q in a given region V is simply,

$$Q = \int_V d^3x \rho(x,t)$$

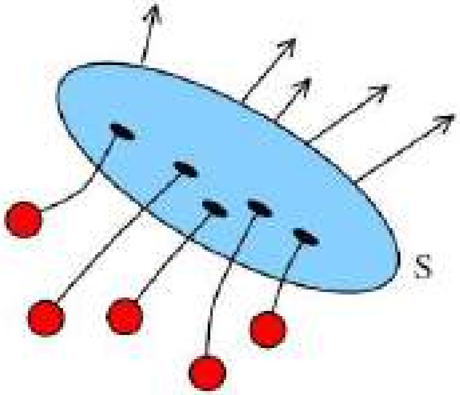
In most situations, we will consider smooth charge densities, which can be thought of as arising from averaging over many point-like particles. But, on occasion, we will return to the idea of a single particle of charge q, moving on some trajectory r(t), by writing $\rho = q\delta(x - r(t))$ where the delta-function ensures that all the charge sits at a point.

More generally, we will need to describe the movement of charge from one place to another. This is captured by a quantity known as the current density J(x, t), defined as follows: for every surface S, the integral

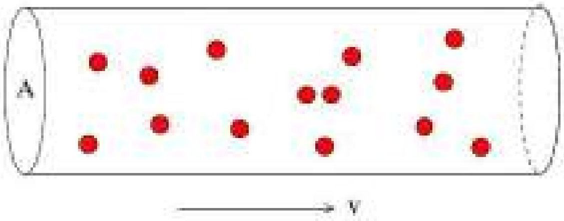
$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

counts the charge per unit time passing through S. (Here dS is the unit normal to S). The quantity I is called the current. In this sense, the current density is the current-per-unit-area. The above is a rather indirect definition of the current density. To get a more intuitive picture, consider a continuous charge distribution in which the velocity of a small volume, at point x, is given by v(x, t). Then, neglecting relativistic effects, the current density is

$$\mathbf{J} = \rho \mathbf{v}$$



In particular, if a single particle is moving with velocity v = ˙r(t), the current density will be J = qvδ3(x − r(t)). This is illustrated in the figure, where the underlying charged particles are shown as red balls, moving through the blue surface S.



As a simple example, consider electrons moving along a wire. We model the wire as a long cylinder of cross-sectional area A as shown below. The electrons move with velocity v, parallel to the axis of the wire. (In reality, the electrons will have some distribution of speeds; we take v to be their average velocity). If there are n electrons per unit volume, each with charge q, then the charge density is ρ = nq and the current density is J = nqv. The current itself is

$$I = |J|A$$

Throughout this course, the current density J plays a much more prominent role than the current I. For this reason, we will often refer to J simply as the current although we'll be more careful with the terminology when there is any possibility for confusion.

1.1.2. The Conservation Law

The most important property of electric charge is that it's conserved. This, of course, means that the total charge in a system can't change. But it means much more than that because electric charge is conserved locally. An electric charge can't just vanish from one part of the Universe and turn up somewhere else. It can only leave one point in space by moving to a neighbouring point.

The property of local conservation means that ρ can change in time only if there is a compensating current flowing into or out of that region. We express this in the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

This is an important equation. It arises in any situation where there is some quantity that is locally conserved.

To see why the continuity equation captures the right physics, it's best to consider the change in the total charge Q contained in some region V.

$$\frac{dQ}{dt} = \int_V d^3x \, \frac{\partial \rho}{\partial t} = -\int_V d^3x \, \nabla \cdot \mathbf{J} = -\int_S \mathbf{J} \cdot d\mathbf{S}$$

From our previous discussion,

$$\int_S J.ds$$

is the total current flowing out through the boundary S of the region V.(It is the total charge flowing out, rather than in, because dS is the outward normal to the region V). The minus sign is there to ensure that if the net flow of current is outwards, then the total charge decreases. If there is no current flowing out of the region, then dQ/dt = 0. This is the statement of (global) conservation of charge. In many applications we will take V to be all of space, R^3, with both charges and currents localised in some compact region. This ensures that the total charge remains constant.

1.1.3. Forces and Fields

Any particle that carries electric charge experiences the force of electromagnetism. But the force does not act directly between particles. Instead, Nature chose to introduce intermediaries. These are fields.

In physics, a "field" is a dynamical quantity which takes a value at every point in space and time. To describe the force of electromagnetism, we need to introduce two fields, each of which is a three-dimensional vector. They are called the electric field E and the magnetic field B,

$$\mathbf{E}(\mathbf{x}, t) \quad \text{and} \quad \mathbf{B}(\mathbf{x}, t)$$

When we talk about a "force" in modern physics, we really mean an intricate interplay between particles and fields. There are two aspects to this. First, the charged particles create both electric and magnetic fields. Second, the electric and magnetic fields guide the charged particles, telling them how to move. This motion, in turn, changes the fields that the particles create. We’re left with a beautiful dance with the particles and fields as two partners, each dictating the moves of the other. This dance between particles and fields provides a paradigm which all other forces in Nature follow. It feels like there should be a deep reason that Nature chose to introduce fields associated to all the forces. And, indeed, this approach does provide one overriding advantage: all interactions are local. Any object — whether particle or field —affects things only in its immediate neighbourhood. This influence can then propagate through the field to reach another point in space, but it does not do so instantaneously. It takes time for a particle in one part of space to influence a particle elsewhere. This lack of instantaneous interaction allows us to introduce forces which are compatible with the theory of special relativity.

The purpose of this course is to provide a mathematical description of the interplay between particles and electromagnetic fields. In fact, you’ve already met one side of this dance: the position r(t) of a particle of charge q is dictated by the electric and magnetic fields through the Lorentz force law,

$$\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$$

The motion of the particle can then be determined through Newton’s equation F = mf.

We explored various solutions to this in the Dynamics and Relativity course. Roughly speaking, an electric field accelerates a particle in the direction E, while a magnetic field causes a particle to move in circles in the plane perpendicular to B. We can also write the Lorentz force law in terms of the charge distribution ρ(x, t) and the current density J(x, t). Now we talk in terms of the force density f (x, t), which is the force acting on a small volume at point x. Now the Lorentz force law reads

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$$

1.1.4. The Maxwell Equations

In this course, most of our attention will focus on the other side of the dance: the way in which electric and magnetic fields are created by charged particles. This is described by a set of four equations, known collectively as the Maxwell equations. They are:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

The equations involve two constants. The first is the electric constant (known also, in slightly old-fashioned terminology, as the permittivity of free space),

$$\epsilon_0 \approx 8.85 \times 10^{-12}\ m^{-3}\, Kg^{-1}\, s^2\, C^2$$

It can be thought of as characterising the strength of the electric interactions. The other is the magnetic constant (or permeability of free space),

$$\begin{aligned} \mu_0 &= 4\pi \times 10^{-7}\ m\, Kg\, C^{-2} \\ &\approx 1.25 \times 10^{-6}\ m\, Kg\, C^{-2} \end{aligned}$$

The presence of 4π in this formula isn't telling us anything deep about Nature. It's more a reflection of the definition of the Coulomb as the unit of charge. Nonetheless, this can be thought of as characterizing the strength of magnetic interactions (in units of Coulombs).

Rather than trying to understand all the equations at once, we'll proceed bit by bit, looking at situations where only some of the equations are important. By the end of the lectures, we will understand the physics captured by each of these equations and how they fit together.

However, equally importantly, we will also explore the mathematical structure of the Maxwell equations. At first glance, they look just like four random equations from vector calculus. Yet this couldn't be further from the truth. The Maxwell equations are special and, when viewed in the right way, are the essentially unique equations that can describe the force of electromagnetism. The full story of why these are the unique equations involves both quantum mechanics and relativity and will only be told in later courses. But we will start that journey here. The goal is that by the end of these lectures you will be convinced of the importance of the Maxwell equations on both experimental and aesthetic grounds.

1.2. Electrostatics:

The electric field produced by stationary source charges is called and electrostatic field. The electric field at a particular point is a vector whose magnitude is proportional to the total force acting on a test charge located at that point, and whose direction is equal to the direction of the force acting on a positive test charge. The electric field E , generated by a collection of source charges, is defined as

$$\overline{E} = \frac{\overline{F}}{Q}$$

where F is the total electric force exerted by the source charges on the test charge Q. It is assumed that the test charge Q is small and therefore does not change the distribution of the source charges. The total force exerted by the source charges on the test charge is equal to

$$\overline{F} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3 + \ldots = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 Q}{r_1^2}\hat{r}_1 + \frac{q_2 Q}{r_2^2}\hat{r}_2 + \frac{q_3 Q}{r_3^2}\hat{r}_3 + \ldots\right) = \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i$$

The electric field generated by the source charges is thus equal to

$$\overline{E} = \frac{\overline{F}}{Q} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i$$

In this section, we will be interested in electric charges at rest. This means that there exists a frame of reference in which there are no currents; only stationary charges. Of course, there will be forces between these charges but we will assume that the charges are pinned in place and cannot move. The question that we want to answer is: what is the electric field generated by these charges? Since nothing moves, we are looking for time independent solutions to Maxwell'sequations with J = 0. This means that we can consistently set B = 0 and we're left with two of Maxwell's equations to solve. They are,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = 0$$

1.2.1. A History of Electrostatics

Perhaps the simplest demonstration of the attractive properties of electric charge comes from rubbing a balloon on your head and sticking it to the wall. This phenomenon was known, at least in spirit, to the ancient Greeks and is credited to Thales of Miletus around 600 BC. Although, in the absence of any ancient balloons, he had to make do with polishing pieces of amber and watching it attract small objects. A systematic, scientific approach to electrostatics starts with William Gilbert, physicist, physician and one-time bursar of St Johns College, Cambridge. (Rumour has it that he'd rather have been at Oxford.) His most important work, De Magnete, published in 1600 showed, among other things, that many materials, not just amber, could be electrified. With due deference, he referred to these as "electrics", derived from the Greek "ηλεκτρον" (electron) meaning "amber". These are materials that we now call "insulators". There was slow progress over the next 150 years, much of it devoted to building machines which could store electricity. A notable breakthrough came from the experiments of the little-known English scientist Stephen Grey, who was the first to appreciate that the difficulty in electrifying certain objects is because they are conductors, with any charge quickly flowing through them and away. Grey spent most of his life as an amateur astronomer, although his amateur status appears to be in large part because he fell foul of Isaac Newton who barred his entry into more professional scientific circles.

He performed his experiments on conductors in the 1720s, late in life when the lack of any income left him destitute and pensioned to Chaterhouse (which was, perhaps, the world's fanciest poorhouse). Upon Newton's death, the scientific community clamoured to make amends. Grey was awarded the Royal Society's first Copley medal. Then, presumably because they felt guilty, he was also awarded the second. Grey's experiments were later reproduced by the French chemist Charles Fran, cois de Cisternay DuFay, who came to the wonderful conclusion that all objects can be electrified by rubbing apart from "metals, liquids and animals". He does not, to my knowledge, state how much rubbing of animals he tried before giving up. He was also the first to notice that static electricity can give rise to both attractive and repulsive forces. By the 1750s, there were many experiments on electricity, but little theory to explain them. Most ideas rested on a fluid description of electricity, but arguments raged over whether a single fluid or two fluids were responsible. The idea that there were both positive and negative charges, then thought of as a surplus and deficit of fluid, was introduced independently by the botanist William Watson and the US founding father Benjamin Franklin. Franklin is arguably the first to suggest that charge is conserved although his statement wasn't quite as concise as the continuity equation:

It is now discovered and demonstrated, both here and in Europe, that the Electrical Fire is a real Element, or Species of Matter, not created by the Friction, but collected only Benjamin Franklin, 1747. Still, it's nice to know that charge is conserved both in the US and in Europe. A quantitative understanding of the theory of electrostatics came only in the 1760s.

A number of people suggested that the electrostatic force follows an inverse-square law, prominent among them Joseph Priestly who is better known for the discovery of Oxygen and, of at least equal importance, the invention of soda water. In 1769, the Scottish physicist John Robison announced that he had measured the force to fall off as $1/r_{2.06}$. This was before the invention of error bars and he seems to receive little credit. Around the same time, the English scientist Henry Cavendish, discover of Hydrogen and weigher of the Earth, performed a number of experiments to demonstrate the inverse-square law but, as with his many of his other electromagnetic discoveries, he chose not to publish. It was left to French physicist Charles Augustin de Coulomb to clean up, publishing the results of his definitive experiments in 1785 on the force that now carries his name.

In its final form, Coulomb's law becomes transmuted into Gauss' law. For once, this was done by the person after whom it's named. Gauss derived this result in 1835, although it wasn't published until 1867.

1.2.2. Gauss' Law

Consider some closed region $V \subset R^3$ of space. We'll denote the boundary of V by $S = \partial V$. By integrating this equation,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

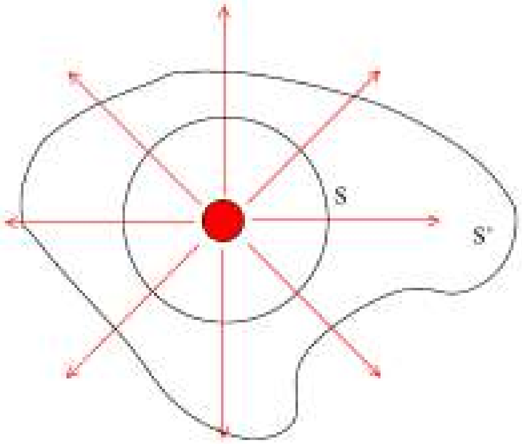
Since the left-hand side is a total derivative, we can use the divergence theorem to convert this to an integral over the surface S. We have,

$$\int_V d^3x \, \nabla \cdot \mathbf{E} = \int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int_V d^3x \, \rho$$

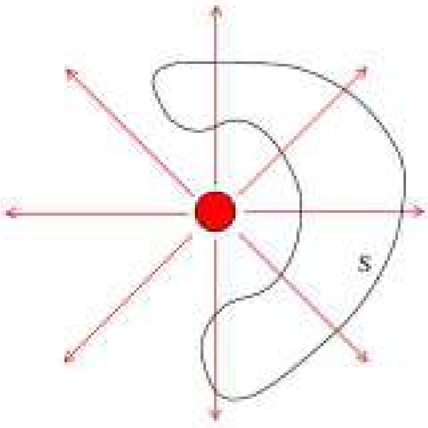
The integral of the charge density over V is simply the total charge contained in the region. We'll call it $Q = \int d^3x \, \rho$. Meanwhile, the integral of the electric field over S is called the flux through S. We learn that the two are related by,

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}$$

This is Gauss's law.



The flux through S and S′ is the same.



The flux through S vanishes.

Notice that it doesn't matter what shape the surface S takes. As long as it surrounds a total charge Q, the flux through the surface will always be Q/ϵ_0. This is shown, for example, in the left-hand figure above. A fancy way of saying this is that the integral of the flux doesn't depend on the geometry of the surface, but does depend on its topology since it must surround the charge Q. The choice of S is called the Gaussian surface; often there's a smart choice that makes a particular problem simple.

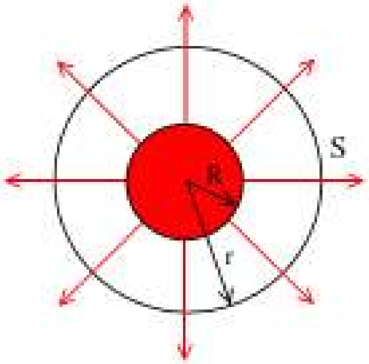
Only charges that lie inside V contribute to the flux. Any charges that lie outside will produce an electric field that penetrates through S at some point, giving negative flux, but leaves through the other side of S, depositing positive flux. The total contribution from these charges that lie outside of V is zero, as illustrated in the right-hand figure above.

1.2.3. The Coulomb Force

Gauss' law reproduces the more familiar Coulomb force law that we all know and love. To do this, take a spherically symmetric charge distribution, centered at the origin, contained within some radius R. This will be our model for a particle. We won't need to make any assumption about the nature of the distribution other than its symmetry and the fact that the total charge is Q.

We want to know the electric field at some radius r >R. We take our Gaussian surface S to be a sphere of radius r as shown in the figure. Gauss' law states,

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}$$



At this point we make use of the spherical symmetry of the problem. This tells us that the electric field must point radially outwards: E(x) = E(r)ˆr. And, since the integral is only over the angular coordinates of the sphere, we can pull the function E(r) outside. We have,

$$\int_S \mathbf{E} \cdot d\mathbf{S} = E(r) \int_S \hat{\mathbf{r}} \cdot d\mathbf{S} = E(r)\, 4\pi r^2 = \frac{Q}{\epsilon_0}$$

where the factor of $4\pi r^2$ has arisen simply because it's the area of the Gaussian sphere. We learn that the electric field outside a spherically symmetric distribution of charge Q is

$$\mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}$$

That's nice. This is the familiar result that we've seen before. The Lorentz force law then tells us that a test charge q moving in the region r > R experiences a force,

$$\mathbf{F} = \frac{Qq}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}$$

This, of course, is the Coulomb force between two static charged particles. Notice that, as promised, $1/\epsilon_0$ characterises the strength of the force. If the two charges have the same sign, so that $Q_q > 0$, the force is repulsive, pushing the test charge away from the origin. If the charges have opposite signs, $Q_q < 0$, the force is attractive, pointing towards the origin.

1.2.4. A Uniform Sphere

The electric field outside a spherically symmetric charge distribution is always given by equation below. What about inside? This depends on the distribution in question. The simplest is a sphere of radius R with uniform charge distribution ρ. The total charge is,

$$Q = \frac{4\pi}{3} R^3 \rho$$

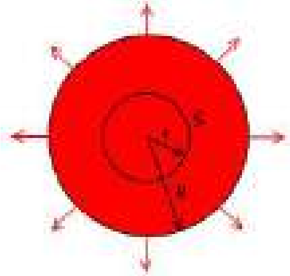
Let's pick our Gaussian surface to be a sphere, centered at the origin, of radius r < R. The charge contained within this sphere is $4\pi\rho r^3/3 = Qr^3/R^3$, so Gauss' law gives,

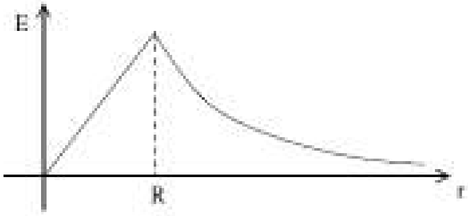
$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Qr^3}{\epsilon_0 R^3}$$

Again, using the symmetry argument we can write E(r) = E(r)ˆr and compute,

$$\int_S \mathbf{E} \cdot d\mathbf{S} = E(r) \int_S \hat{\mathbf{r}} \cdot d\mathbf{S} = E(r) 4\pi r^2 = \frac{Qr^3}{\epsilon_0 R^3}$$

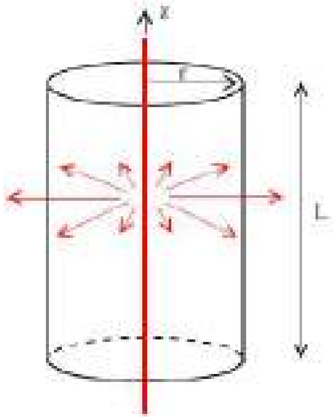
This tells us that the electric field grows linearly inside the sphere. At the surface of the sphere, r = R, the electric field is continuous but the derivative, dE/dr, is not. This is shown in the graph below,





1.2.5. Line Charges

Consider, next, a charge smeared out along a line which we'll take to be the z-axis. We'll take uniform charge density η per unit length. (If you like you could consider a solid cylinder with uniform charge density and then send the radius to zero).



We want to know the electric field due to this line of charge. Our set-up now has cylindrical symmetry. We take the Gaussian surface to be a cylinder of length L and radius r. We have,

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{\eta L}{\epsilon_0}$$

Again, by symmetry, the electric field points in the radial direction, away from the line. We'll denote this vector in cylindrical polar coordinates as ˆr so that E = E(r)ˆr. The symmetry means that the two end caps of the Gaussian surface don't contribute to the integral because their normal points in the ˆz direction and ˆz · ˆr = 0. We're left only with a contribution from the curved side of the cylinder,

$$\int_S \mathbf{E} \cdot d\mathbf{S} = E(r)\, 2\pi r L = \frac{\eta L}{\epsilon_0}$$

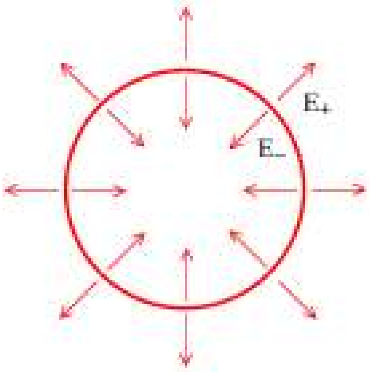
So that the electric field is,

$$\mathbf{E}(r) = \frac{\eta}{2\pi\epsilon_0 r}\hat{\mathbf{r}}$$

Note that, while the electric field for a point charge drops off as $1/r^2$ (with r the radial distance), the electric field for a line charge drops off more slowly as 1/r. (Of course, the radial distance r means slightly different things in the two cases: it is $r = \text{sqrt}(x^2 + y^2 + z^2)$ for the point particle, but is $r = \text{sqrt}(x^2 + y^2)$ for the line).

1.2.6. A Spherical Shell

Let's give one last example that involves surface charge and the associated discontinuity of the electric field. We'll consider a spherical shell of radius R, centered at the origin, with uniform surface charge density σ.



The total charge is,

$$Q = 4\pi R^2 \sigma$$

We already know that outside the shell, r > R, the electric field takes the standard inverse-square form. What about inside? Well, since any surface with r < R doesn't surround a charge, Gauss' law tells us that we necessarily have E = 0 inside. That means that there is a discontinuity at the surface r = R,

$$\mathbf{E} \cdot \hat{\mathbf{r}}\big|_{+} - \mathbf{E} \cdot \hat{\mathbf{r}}\big|_{-} = \frac{Q}{4\pi R^2 \epsilon_0} = \frac{\sigma}{\epsilon_0}$$

1.3. The Electrostatic Potential

For all the examples in the last section, symmetry considerations meant that we only needed to consider Gauss' law. However, for general charge distributions Gauss' law is not sufficient. We also need to invoke the second equation, ∇ × E = 0. In fact, this second equation is easily dispatched since ∇ × E = 0 implies that the electric field can be written as the gradient of some function,

$$\mathbf{E} = -\nabla \phi$$

The scalar ϕ is called the electrostatic potential or scalar potential (or, sometimes, just the potential). To proceed, we revert to the original differential form of Gauss' law. This now takes the form of the Poisson equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \Rightarrow \quad \nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

In regions of space where the charge density vanishes, we're left solving the Laplace equation

$$\nabla^2 \phi = 0$$

A few comments:

- The potential ϕ is only defined up to the addition of some constant. This seemingly trivial point is actually the beginning of a long and deep story in theoretical physics known as gauge invariance. For now, we'll eliminate this redundancy by requiring that ϕ(r) → 0 as r → ∞.

- We know from our study of Newtonian mechanics that the electrostatic potential is proportional to the potential energy experienced by a test particle. Specifically, a test particle of mass m, position r(t) and charge q moving in a background electric field has conserved energy

$$E = \frac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + q\phi(\mathbf{r})$$

1.3.1. The Point Charge

Let’s start by deriving the Coulomb force law yet again. We’ll take a particle of charge Q and place it at the origin. This time, however, we’ll assume that the particle really is a point charge. This means that the charge density takes the form of a delta-function, ρ(x) = Qδ^3(x). We need to solve the equation,

$$\nabla^2\phi = -\frac{Q}{\epsilon_0}\delta^3(\mathbf{x})$$

1.3.2. The Point Charge

Let’s start by deriving the Coulomb force law yet again. We’ll take a particle of charge Q and place it at the origin. This time, however, we’ll assume that the particle really is a point charge. This means that the charge density takes the form of a delta-function, ρ(x) = Qδ^3(x). We need to solve the equation,

$$\nabla^2\phi = -\frac{Q}{\epsilon_0}\delta^3(\mathbf{x})$$

Let’s recall how we find this solution. We first look away from the origin, $r \neq 0$, where there’s no funny business going on with delta-function. Here, we’re looking for the spherically symmetric solution to the Laplace equation. This is

$$\phi = \frac{\alpha}{r}$$

for some constant α. To see why this solves the Laplace equation, we need to use the result

$$\nabla r = \hat{\mathbf{r}}$$

where ˆr is the unit radial vector in spherical polar coordinates, so x = rˆr. Using the chain rule, this means that ∇(1/r) = −ˆr/r^2 = −x/r^3. This gives us,

$$\nabla\phi = -\frac{\alpha}{r^3}\mathbf{x} \quad \Rightarrow \quad \nabla^2\phi = -\alpha\left(\frac{\nabla \cdot \mathbf{x}}{r^3} - \frac{3\mathbf{x} \cdot \mathbf{x}}{r^5}\right)$$

But ∇ · x = 3 and we find that ∇2ϕ = 0 as required.

It remains to figure out what to do at the origin where the delta-function lives. This is what determines the overall normalization α of the solution. At this point, it’s simplest to use the integral form of Gauss’ law to transfer the problem from the origin to the far flung reaches of space.

$$\rho(\mathbf{x}) = Q\,\delta^3(\mathbf{x}) \quad \Rightarrow \quad \int_V d^3x\,\rho = Q$$

So, the Guass law becomes

$$\int_S \nabla\phi \cdot d\mathbf{S} = -\frac{Q}{\epsilon_0}$$

Substituting $\phi = \alpha/r$ into the above equation, and choosing S to be a sphere of radius r, tells us that we must have $\alpha = Q/4\pi\epsilon_0$

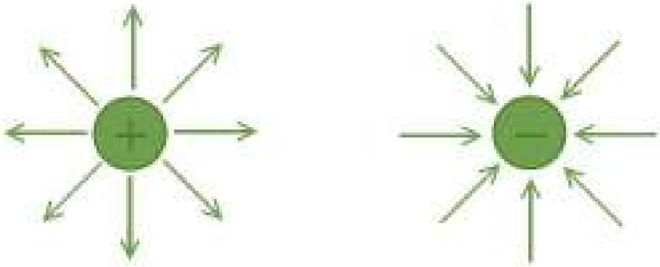
Taking the gradient of $\alpha = Q/4\pi\epsilon_0$ gives us Coulomb's law,

$$\mathbf{E}(\mathbf{x}) = -\nabla\phi = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}$$

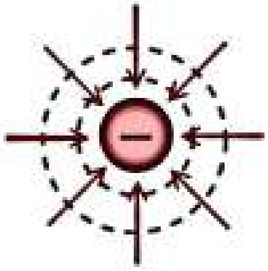
The derivation of Coulomb's law using the potential was somewhat more involved than the technique using Gauss' law alone. However, as we'll now see, introducing the potential allows us to write down the solution to essentially any problem.

1.3.3. Field Lines

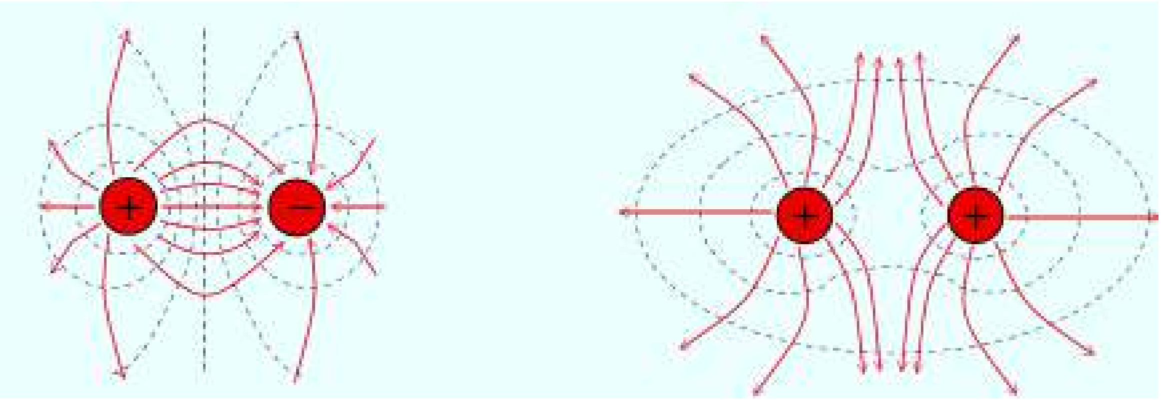
The usual way of depicting a vector is to draw an arrow whose length is proportional to the magnitude. For the electric field, there's a slightly different, more useful way to show what's going on. We draw continuous lines, tangent to the electric field E, with the density of lines proportional to the magnitude of E. This innovation, due to Faraday, is called the field line. Field lines are continuous. They begin and end only at charges. They can never cross. The field lines for positive and negative point charges are:



By convention, the positive charges act as sources for the lines, with the arrows emerging. The negative charges act as sinks, with the arrows approaching. It's also easy to draw the equipotentials — surfaces of constant ϕ — on this same figure below. These are the surfaces along which you can move a charge without doing any work. The relationship $E = -\nabla\phi$ ensures that the equipotentials cut the field lines at right angles. We usually draw them as dotted lines:



Meanwhile, we can (very) roughly sketch the field lines and equipotentials for the dipole (on the left) and for a pair of charges of the same sign (on the right):



1.3.4. Electrostatic Equilibrium

Here's a simple question: can you trap an electric charge using only other charges? In other words, can you find some arrangements of charges such that a test charge sits in stable equilibrium, trapped by the fields of the others. There's a trivial way to do this: just allow a negative charge to sit directly on top of a positive charge. But let's throw out this possibility. We'll ask that the equilibrium point lies away from all the other charges.

There are some simple set-ups that spring to mind that might achieve this. Maybe you could place four positive charges at the vertices of a pyramid; or perhaps 8 positive charges at the corners of a cube. Is it possible that a test positive charge trapped in the middle will be stable? It's certainly repelled from all the corners, so it might seem plausible.

The answer, however, is no. There is no electrostatic equilibrium. You cannot trap an electric charge using only other stationary electric charges, at least not in a stable manner. Since the potential energy of the particle is proportional to ϕ, mathematically, this is the statement that a harmonic function, obeying $\nabla^2\phi = 0$, can have no minimum or maximum.

To prove that there can be no electrostatic equilibrium, let's suppose the opposite: that there is some point in empty space $r_\star$ that is stable for a particle of charge $q > 0$. By "empty space", we mean that $\rho(r) = 0$ in a neighbourhood of $r_\star$. Because the point is stable, if the particle moves away from this point then it must always be pushed back. This, in turn, means that the electric field must always point inwards towards the point $r_\star$; never away. We could then surround $r_\star$ by a small surface S and compute,

$$\int_S \mathbf{E} \cdot d\mathbf{S} < 0$$

But, by Gauss' law, the right-hand side must be the charge contained within S which, by assumption, is zero. This is our contradiction: electrostatic equilibrium does not exist.

Of course, if you're willing to use something other than electrostatic forces then you can construct equilibrium situations. For example, if you restrict the test particle to lie on a plane then it's simple to check that equal charges placed at the corners of a polygon will result in a stable equilibrium point in the middle. But to do this you need to use other forces to keep the particle in the plane in the first place.

1.3.5. Electrostatic Energy

There is energy stored in the electric field. In this section, we calculate how much. Let's start by recalling a fact from our first course on classical mechanics1. Suppose we have some test charge q moving in a background electrostatic potential ϕ. We'll denote the potential energy of the particle as U(r). The potential U(r) of the particle can be thought of as the work done bringing the particle in from infinity;

$$U(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = +q \int_{\infty}^{\mathbf{r}} \nabla\phi \cdot d\mathbf{r} = q\phi(\mathbf{r})$$

where we've assumed our standard normalization of $\phi(r) \to 0$ as $r \to \infty$.

Consider a distribution of charges which, for now, we'll take to be made of point charges qi at positions r_i. The electrostatic potential energy stored in this configuration is the same as the work required to assemble the configuration in the first place.

So how much work does it take to assemble a collection of charges?

Well, the first charge is free. In the absence of any electric field, you can just put it where you like — say, r1. The work required is $W_1 = 0$. To place the second charge at r2 takes work,

$$W_2 = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

Note that if the two charges have the same sign, so $q1q2 > 0$, then $W2 > 0$ which is telling us that we need to put work in to make them approach. If $q1q2 < 0$ then $W2 < 0$ where the negative work means that the particles wanted to be drawn closer by their mutual attraction.

The third charge has to battle against the electric field due to both q_1 and q_2. The work required is

$$W_3 = \frac{q_3}{4\pi\epsilon_0} \left(\frac{q_2}{|\mathbf{r}_2 - \mathbf{r}_3|} + \frac{q_1}{|\mathbf{r}_1 - \mathbf{r}_3|} \right)$$

and so on. The total work needed to assemble all the charges is the potential energy stored in the configuration,

$$U = \sum_{i=1}^{N} W_i = \frac{1}{4\pi\epsilon_0} \sum_{i<j} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

Where ∑ i < j means that we sum over each pair of particles once. In fact, you probably could have just written down as the potential energy stored in the configuration. The whole purpose of the above argument was really just to nail down a factor of 1/2:do we sum over all pairs of particles ∑ i < j or all particles. The answer, as we have seen, is all pairs.

We can make that factor of 1/2 even more explicit by writing,

$$U = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i} \sum_{j\neq i} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

where now we sum over each pair twice. The potential at ri due to all the other charges qj , j≠ i is,

$$\phi(\mathbf{r}_i) = \frac{1}{4\pi\epsilon_0} \sum_{j\neq i} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

which means that we can write the potential energy as

$$U = \frac{1}{2} \sum_{i=1}^{N} q_i \phi(\mathbf{r}_i)$$

This is the potential energy for a set of point charges. But there is an obvious generalization to charge distributions ρ(r). We'll again assume that ρ(r) has compact support so that the charge is localised in some region of space. The potential energy associated to such a charge distribution should be

$$U = \frac{1}{2} \int d^3r \; \rho(\mathbf{r}) \, \phi(\mathbf{r})$$

where we can quite happily take the integral over all of R3, safe in the knowledge that anywhere that doesn't contain charge has ρ(r) = 0 and so won't contribute. Now this is in a form that we can start to play with. We use Gauss'slaw to rewrite it as,

$$U = \frac{\epsilon_0}{2} \int d^3r \; (\nabla \cdot \mathbf{E}) \, \phi = \frac{\epsilon_0}{2} \int d^3r \, [\nabla \cdot (\mathbf{E}\phi) - \mathbf{E} \cdot \nabla\phi]$$

But the first term is a total derivative. And since we're taking the integral over all of space and ϕ(r) → 0 as r → ∞, this term just vanishes. In the second term we can replace ∇ϕ = −E. We find that the potential energy stored in a charge distribution has an elegant expression solely in terms of the electric field that it creates,

$$U = \frac{\epsilon_0}{2}\int d^3r\ \mathbf{E}\cdot\mathbf{E}$$

A Quick Foray into Quantum Electrodynamics

To assign a meaning of "radius" to seemingly point-like particles, we really need the machinery of quantum field theory. In that context, the size of the electron is called its Compton wavelength. This is the distance scale at which the electron gets surrounded by a swarm of electron-positron pairs which, roughly speaking, smears out the charge distribution. This distance scale is

$$a = \frac{\hbar}{m_e c}$$

For the whole story to hang together, we require

$$\frac{e^2}{8\pi\epsilon_0\hbar c} < 1$$

This is an almost famous combination of constants. It's more usual to define the combination,

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$$

This is known as the fine structure constant.

1.4. The Force Between Electric Dipoles

As an application of our formula for electrostatic energy, we can compute the force between two, far separated dipoles. We place the first dipole, p1, at the origin. It gives rise to a potential,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}_1\cdot\mathbf{r}}{r^3}$$

Now, at some distance away, we place a second dipole. We'll take this to consist of a charge Q at position r and a charge −Q at position r − d, with d ≪ r. The resulting dipole moment is p2 = Qd. We're not interested in the energy stored in each individual dipole; only in the potential energy needed to bring the two dipoles together. This is given by,

$$
\begin{aligned}
U = Q\Big(\phi(\mathbf{r}) - \phi(\mathbf{r}-\mathbf{d})\Big) &= \frac{Q}{4\pi\epsilon_0}\left(\frac{\mathbf{p}_1\cdot\mathbf{r}}{r^3} - \frac{\mathbf{p}_1\cdot(\mathbf{r}-\mathbf{d})}{|\mathbf{r}-\mathbf{d}|^3}\right) \\
&= \frac{Q}{4\pi\epsilon_0}\left(\frac{\mathbf{p}_1\cdot\mathbf{r}}{r^3} - \mathbf{p}_1\cdot(\mathbf{r}-\mathbf{d})\left(\frac{1}{r^3} + \frac{3\mathbf{d}\cdot\mathbf{r}}{r^5} + \ldots\right)\right) \\
&= \frac{Q}{4\pi\epsilon_0}\left(\frac{\mathbf{p}_1\cdot\mathbf{d}}{r^3} - \frac{3(\mathbf{p}_1\cdot\mathbf{r})(\mathbf{d}\cdot\mathbf{r})}{r^5}\right)
\end{aligned}
$$

where, to get to the second line, we've Taylor expanded the denominator of the second term. This final expression can be written in terms of the second dipole moment. We find the nice, symmetric expression for the potential energy of two dipoles separated by distance r,

$$U = \frac{1}{4\pi\epsilon_0}\left(\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{r^3} - \frac{3(\mathbf{p}_1\cdot\mathbf{r})(\mathbf{p}_2\cdot\mathbf{r})}{r^5}\right)$$

But, we know from our first course on dynamics that the force between two objects is just given by F = −∇U. We learn that the force between two dipoles is given by,

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\nabla\left(\frac{3(\mathbf{p}_1\cdot\mathbf{r})(\mathbf{p}_2\cdot\mathbf{r})}{r^5} - \frac{\mathbf{p}_1\cdot\mathbf{p}_2}{r^3}\right)$$

The strength of the force, and even its sign, depends on the orientation of the two dipoles. If p_1 and p_2 lie parallel to each other and to r then the resulting force is attractive. If p1 and p2 point in opposite directions, and lie parallel to r, then the force is repulsive. The expression above allows us to compute the general force.

1.5. Magnetostatics

Charges give rise to electric fields. Current give rise to magnetic fields. In this section, we will study the magnetic fields induced by steady currents. This means that we are again looking for time independent solutions to the Maxwell equations. We will also restrict to situations in which the charge density vanishes, so ρ = 0. We can then set E = 0 and focus our attention only on the magnetic field. We're left with two Maxwell equations to solve:

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}$$

$$\nabla\cdot\mathbf{B} = 0$$

If you fix the current density J, these equations have a unique solution.

1.5.1. Steady Currents

Because ρ = 0, there can't be any net charge. But, of course, we still want charge to be moving! This means that we necessarily have both positive

and negative charges which balance out at all points in space. Nonetheless, these charges can move so there is a current even though there is no net charge transport. This may sound artificial, but in fact it's exactly what happens in a typical wire. In that case, there is background of positive charge due to the lattice of ions in the metal. Meanwhile, the electrons are free to move. But they all move together so that at each point we still have ρ = 0. The continuity equation, which captures the conservation of electric charge, is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Since the charge density is unchanging (and, indeed, vanishing), we have

$$\nabla \cdot \mathbf{J} = 0$$

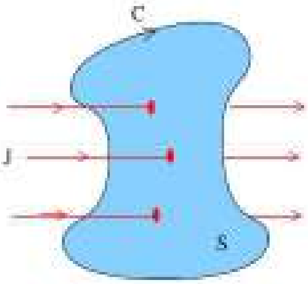
Mathematically, this is just saying that if a current flows into some region of space, an equal current must flow out to avoid the build up of charge. Note that this is consistent with since, for any vector field, ∇ · (∇ × B) = 0.

1.5.2. Ampere's Law

The first equation of magnetostatics,

$$\oint \overrightarrow{B} \cdot \overrightarrow{dl} = \mu_o N I$$

is known as Ampere's law. As with many of these vector differential equations, there is an equivalent form in terms of integrals. In this case, we choose some open surface S with boundary C = ∂S. Integrating over the surface, we can use Stokes' theorem to turn the integral of ∇ × B into a line integral over the boundary C,



$$\int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}$$

Recall that there's an implicit orientation in these equations. The surface S comes with a normal vector ˆn which points away from S in one direction. The line integral around the boundary is then done in the right-handed sense, meaning that if you stick the thumb of your right hand in the direction ˆn then your fingers curl in the direction of the line integral.

The integral of the current density over the surface S is the same thing as the total current I that passes through S. Ampere’s law in integral form then reads

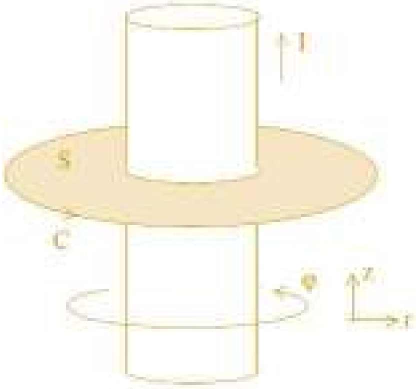
$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

1.5.3. A Long Straight Wire

Consider an infinite, straight wire carrying current I. We’ll take it to point in the ˆz direction. The symmetry of the problem is jumping up and down telling us that we need to use cylindrical polar coordinates, (r, φ, z), where r = sqrt(x2 + y2) is the radial distance away from the wire.

We take the open surface S to lie in the x − y plane, centered on the wire. For the line integral to give something that doesn’t vanish, it’s clear that the magnetic field has to have some component that lies along the circumference of the disc.

But, by the symmetry of the problem, that’s actually the only component that B can have: it must be of the form B = B(r) ˆφ. (If this was a bit too quick, we’ll derive this more carefully below). Any magnetic field of this form automatically satisfies the second Maxwell equation ∇ ▪ B = 0. We need only worry about Ampere’s law which tells us



$$\oint_C \mathbf{B} \cdot d\mathbf{r} = B(r) \int_0^{2\pi} r\, d\varphi = 2\pi r B(r) = \mu_0 I$$

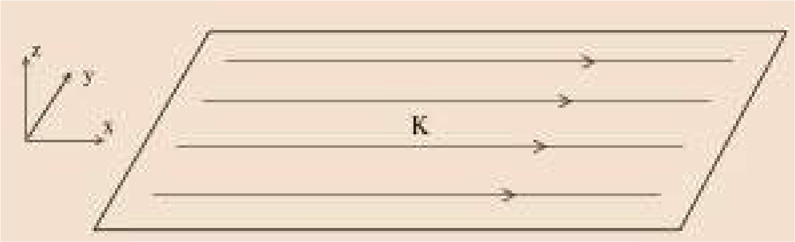
We see that the strength of the magnetic field is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\boldsymbol{\varphi}}$$

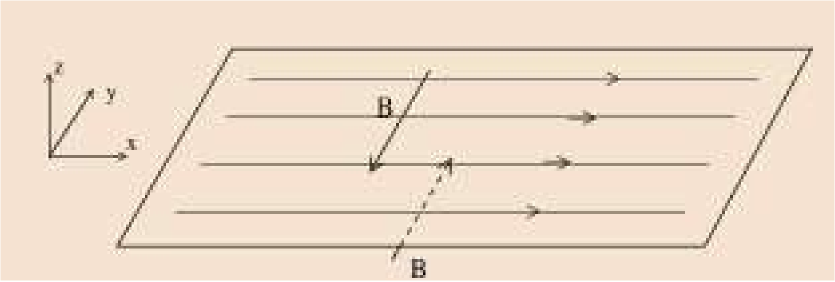
The magnetic field circles the wire using the ”right-hand rule”: stick the thumb of your right hand in the direction of the current and your fingers curl in the direction of the magnetic field. Note that the simplest example of a magnetic field falls off as 1/r. In contrast, the simplest example of an electric field – the point charge – falls of as $1/r^2$. You can trace this difference back to the geometry of the two situations. Because magnetic fields are sourced by currents, the simplest example is a straight line and the 1/r fall-off is because there are two transverse directions to the wire. When we look at a line of charge, the electric field also drops off as 1/r.

1.5.4. Surface Currents and Discontinuities

Consider the flat plane lying at z = 0 with a surface current density that we'll call K. Note that K is the current per unit length, as opposed to J which is the current per unit area. You can think of the surface current as a bunch of wires, all lying parallel to each other. We'll take the current to lie in the x-direction: K = Kˆx as shown below.



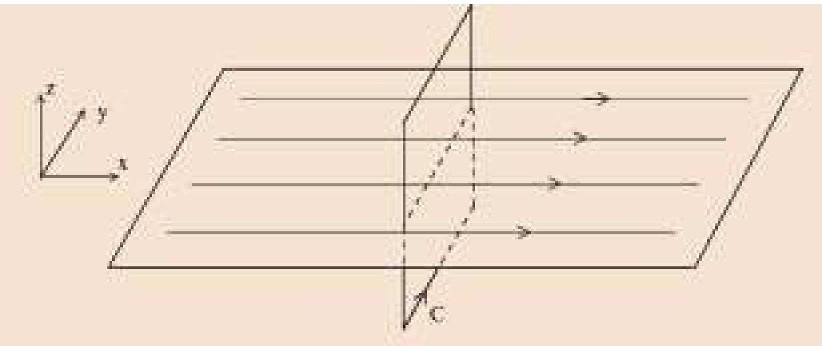
From our previous result, we know that the B field should curl around the current in the right-handed sense. But, with an infinite number of wires, this can only mean that B is oriented along the y direction. In fact, from the symmetry of the problem, it must look like



with B pointing in the −ˆy direction when z > 0 and in the +ˆy direction when z < 0. We write,

$$\mathbf{B} = -B(z)\hat{\mathbf{y}}$$

with B(z) = −B(−z). We invoke Ampere's law using the following open surface:



with length L in the y direction and extending to +-z. We have,

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = LB(z) - LB(-z) = 2LB(z) = \mu_0 KL$$

so we find that the magnetic field is constant above an infinite plane of surface current,

$$B(z) = \frac{\mu_0 K}{2} \qquad z > 0$$

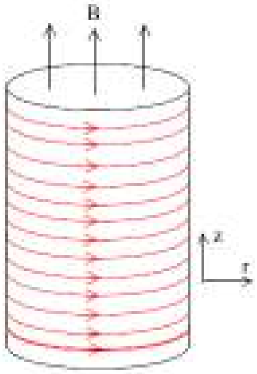
This is rather similar to the case of the electric field in the presence of an infinite plane of surface charge.

1.5.5. A Solenoid

A solenoid consists of a surface current that travels around a cylinder. It’s simplest to think of a single current-carrying wire winding many times around the outside of the cylinder. (Strictly speaking, the cross-sectional shape of the solenoid doesn’t have to be a circle – it can be anything. But we’ll stick with a circle here for simplicity). To make life easy, we’ll assume that the cylinder is infinitely long. This just means that we can neglect effects due to the ends.

We’ll again use cylindrical polar coordinates, (r, φ, z), with the axis of the cylinder along ˆz. By symmetry, we know that B will point along the z-axis. Its magnitude can depend only on the radial distance: B = B(r)ˆz. Once again, any magnetic field of this form immediately satisfies

∇ · B = 0.

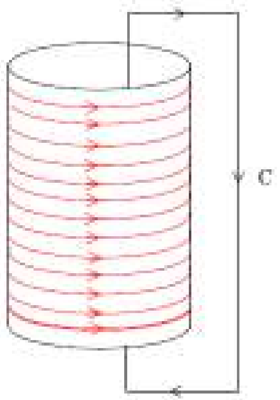


We solve Ampere’s law in differential form. Anywhere other than the surface of the solenoid, we have J = 0 and

$$\nabla \times \mathbf{B} = 0 \quad \Rightarrow \quad \frac{dB}{dr} = 0 \quad \Rightarrow \quad B(r) = \text{constant}$$

Outside the solenoid, we must have B(r) = 0 since B(r) is constant and we know B(r) → 0 as r → ∞. To figure out the magnetic field inside the solenoid, we turn to the integral form of Amp`ere's law and consider the surface S, bounded by the curve C shown in the figure. Only the line that runs inside the solenoid contributes to the line integral. We have,

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = BL = \mu_0 INL$$



where N is the number of windings of wire per unit length. We learn that inside the solenoid, the constant magnetic field is given by,

$$\mathbf{B} = \mu_0 IN\,\hat{\mathbf{z}}$$

1.5.6. The Vector Potential

For the simple current distributions of the last section, symmetry considerations were enough to lead us to a magnetic field which automatically satisfied

$$\nabla \cdot \mathbf{B} = 0$$

But, for more general currents, this won't be the case. Instead we have to ensure that the second magnetostatic Maxwell equation is also satisfied.

In fact, this is simple to do. We are guaranteed a solution to $\nabla \cdot B = 0$ if we write the magnetic field as the curl of some vector field,

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Here A is called the vector potential. While magnetic fields that can be written in the form certainly satisfy $\nabla \cdot B = 0$, the converse is also true; any divergence-free magnetic field can be written as for some A.

(Actually, this previous sentence is only true if our space has a suitably simple topology. Since we nearly always think of space as R3 or some open ball on R3, we rarely run into subtleties. But if space becomes more interesting then the possible solutions to ∇ · B = 0 also become more interesting).

Ampere’s law becomes,

$$\nabla \times \mathbf{B} = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = \mu_0 \mathbf{J}$$

where, in the first equality, we’ve used a standard identity from Vector Calculus. This is the equation that we have to solve to determine A and, through that, B.

- **Something a Little Misleading: The Magnetic Scalar Potential**

There is another quantity that is sometimes used called the magnetic scalar potential, Ω. The idea behind this potential is that you might be interested in computing the magnetic field in a region where there are no currents and the electric field is not changing with time. In this case, you need to solve ∇ × B = 0, which you can do by writing

$$B = -\nabla\Omega$$

Now calculations involving the magnetic field really do look identical to those involving the electric field.

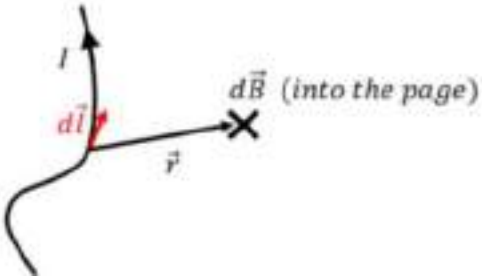
However, you should be wary of writing the magnetic field in this way. We can always solve two of Maxwell’s equations by writing E and B in terms of the electric potential ϕ and vector potential A and this formulation becomes important as we move onto more advanced areas of physics. In contrast, writing B = −∇Ω is only useful in a limited number of situations. The reason for this really gets to the heart of the difference between electric and magnetic fields: electric charges exist; magnetic charges don’t!

1.6. Bio-Savart Law:

The Biot-Savart law allows us to determine the magnetic field at some position in space that is due to an electric current. More precisely, the Biot-Savart law allows us to calculate the infinitesimal magnetic field, $\underset{dB}{\rightarrow}$, that is produced by a small section of wire, $\underset{dl}{\rightarrow}$, carrying current, II, such that $\underset{dl}{\rightarrow}$ is co-linear with the wire and points in the direction of the electric current:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\vec{dl} \times \hat{r}}{r^2}$$

where, $r^{\rightarrow}$, is the vector from the element of wire, $dl^{\rightarrow}$, to the point where we would like to determine the magnetic field, as illustrated in Figure below. μ_0 is a constant of proportionality called the “permeability of free space”, and has the value $\mu_0 = 4\pi \times 10^{-7}$ T·m/A.



We're now going to use the vector potential to solve for the magnetic field B in the presence of a general current distribution. From now, we'll always assume that we're working in Coulomb gauge and our vector potential obeys ∇ · A = 0. Then Ampere's law becomes a whole lot easier: we just have to solve,

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

But this is just something that we've seen already. To see why, it's perhaps best to write it out in Cartesian coordinates. This then becomes three equations,

$$\nabla^2 A_i = -\mu_0 J_i \quad (i = 1, 2, 3)$$

and each of these is the Poisson equation.

It's worth giving a word of warning at this point: the expression ∇^2A is simple in Cartesian coordinates where, as we've seen above, it reduces to the Laplacian on each component. But, in other coordinate systems, this is no longer true. The Laplacian now also acts on the basis vectors such as ˆr and ˆφ. So in these other coordinate systems, ∇^2A is a little more of a mess. (You should probably use the identity ∇^2A = −∇ × (∇ × A) + ∇(∇ .A) if you really want to compute in these other coordinate systems).

Anyway, if we stick to Cartesian coordinates then everything is simple. We know how to write down the most general solution using Green's functions. It is

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V d^3x' \, \frac{J_i(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

Or, if you're feeling bold, you can revert back to vector notation and write

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

1.7. The Magnetic Field

Simply we can calculate and compute the magnetic field B = ∇×A. Again, we need to remember that the ∇ acts on the x rather than the x′. We find

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V d^3x' \, \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}$$

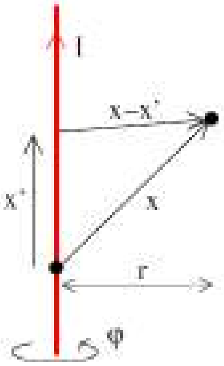
This is known as the Biot-Savart law, which was discussed earlier. It describes the magnetic field due to a general current density.

There is a slight variation on above equation of B(x) which more often goes by the name of the Biot- Savart law. This arises if the current is restricted to a thin wire which traces out a curve C. Then, for a current density J passing through a small volume δV , we write JδV = (JA)δx where A is the cross sectional area of the wire and δx lies tangent to C. Assuming that the cross-sectional area is constant throughout the wire, the current I = JA is also constant. The Biot-Savart law becomes,

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_C \frac{d\mathbf{x}' \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}$$

1.7.1.An Example: The Straight Wire Revisited

Of course, we already derived the answer for a straight wire in without using this fancy vector potential technology. Before proceeding, we should quickly check that the Biot-Savart law reproduces our earlier result. As before, we'll work in cylindrical polar,



coordinates. We take the wire to point along the ˆz axis and use $r^2 = x^2 + y^2$ as our radial coordinate. This means that the line element along the wire is parametrised by dx′=ˆzdz and, for a point x away from the wire, the vector dx′×(x−x′) points along the tangent to the circle of radius r,

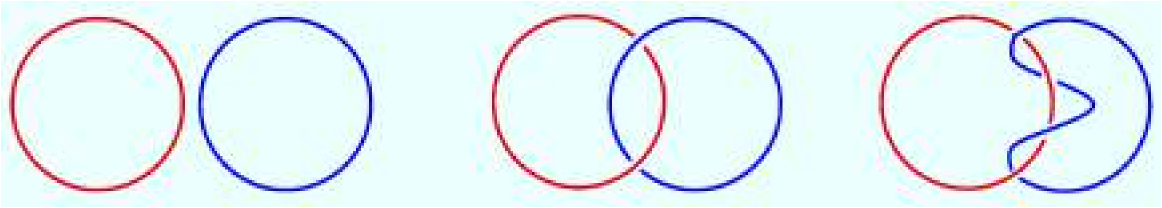
$$d\mathbf{x}' \times (\mathbf{x} - \mathbf{x}') = r\hat{\varphi}\, dz$$

So we have

$$\mathbf{B} = \frac{\mu_0 I \hat{\varphi}}{4\pi} \int_{-\infty}^{+\infty} dz\, \frac{r}{(r^2 + z^2)^{3/2}} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$$

1.7.2. A Mathematical Diversion: The Linking Number

There's a rather cute application of these ideas to pure mathematics. Consider two closed, non-intersecting curves, C and C′, in R^3. For each pair of curves, there is an integer n ∈ Z called the linking number which tells you how many times one of the curves winds around the other. For example, here are pairs of curves with linking number |n| = 0, 1 and 2.



Curves with linking number $n = 0$, $n = 1$ and $n = 2$.

To determine the sign of the linking number, we need to specify the orientation of each curve. In the last two figures above, the linking numbers are negative, if we traverse both red and blue curves in the same direction. The linking numbers are positive if we traverse one curve in a clockwise direction, and the other in an anti-clockwise direction. Importantly, the linking number doesn't change as you deform either curve, provided that the two curves never cross. In fancy language, the linking number is an example of a topological invariant. There is an integral expression for the linking number, first written down by Gauss during his exploration of electromagnetism. The Biot-Savart formula offers a simple physics derivation of Gauss' expression. Suppose that the curve C carries a current I. This sets us a magnetic field everywhere in space.

We will then compute

$$\oint_{C'} \mathbf{B} \cdot d\mathbf{x}'$$

around another curve C. (If you want a justification for computing then you can think of it as the work done when transporting a magnetic monopole of unit charge around C, but this interpretation isn't necessary for what follows.) The Biot-Savart formula gives,

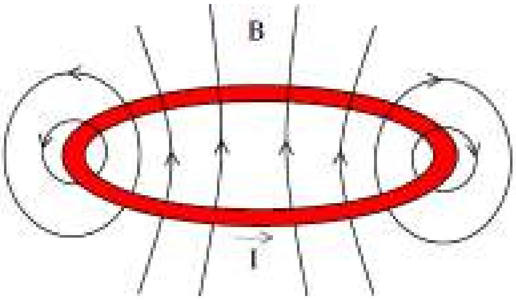
$$\oint_{C'} \mathbf{B}(\mathbf{x}') \cdot d\mathbf{x}' = \frac{\mu_0 I}{4\pi} \oint_{C'} d\mathbf{x}' \cdot \oint_C \frac{d\mathbf{x} \times (\mathbf{x}' - \mathbf{x})}{|\mathbf{x} - \mathbf{x}'|^3}$$

1.8. Magnetic Dipoles

We've seen that the Maxwell equations forbid magnetic monopoles with a long-range $B \sim 1/r^2$ fall-off. So what is the generic fall-off for some distribution of currents which are localised in a region of space? In this section we will see that, if you're standing suitably far from the currents, you'll typically observe a dipole-like magnetic field.

1.8.1. A Current Loop

We start with a specific, simple example. Consider a circular loop of wire C of radius R carrying a current I. We can guess what the magnetic field looks like simply by patching together our result for straight wires: it must roughly take the shape shown in the figure However, we can be more accurate. Here we restrict ourselves only to the magnetic field far from the loop.



To compute the magnetic field far away, we won't start with the Biot-Savart law but instead return to the original expression for A. We're going to return to the notation in which a point in space is labelled as r rather than x. (This is more appropriate for long-distance distance fields which are essentially an expansion in r = |r|). The vector potential is then given by,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Writing this in terms of the current I (rather than the current density J), we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint_C \frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

We want to ask what this looks like far from the loop. Just as we did for the electrostatic potential, we can Taylor expand the integrand using,

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \dots$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint_C d\mathbf{r}' \left(\frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \dots\right)$$

The first term in this expansion vanishes because we're integrating around a circle. This is just a reflection of the fact that there are no magnetic monopoles. For the second term, there's a way to write it in slightly more manageable form. To see this, let's introduce an arbitrary constant vector g and use this to look at,

$$\oint_C d\mathbf{r}' \cdot \mathbf{g}\,(\mathbf{r}\cdot\mathbf{r}')$$

Recall that, from the point of view of this integral, both g and r are constant vectors; it's the vector r′ that we're integrating over. This is now the kind of line integral of a vector that allows us to use Stokes' theorem. We have,

$$\oint_C d\mathbf{r}' \cdot \mathbf{g}\,(\mathbf{r}\cdot\mathbf{r}') = \int_S d\mathbf{S}\cdot\nabla\times(\mathbf{g}\,(\mathbf{r}\cdot\mathbf{r}')) = \int_S dS_i\,\epsilon_{ijk}\partial'_j(g_k r_l r'_l)$$

where, in the final equality, we've resorted to index notation to help us remember what's connected to what. Now the derivative ∂′ acts only on the r′ and we get

$$\oint_C d\mathbf{r}' \cdot \mathbf{g}\,(\mathbf{r}\cdot\mathbf{r}') = \int_S dS_i\,\epsilon_{ijk} g_k r_j = \mathbf{g}\cdot\int_S d\mathbf{S}\times\mathbf{r}$$

But this is true for all constant vectors g which means that it must also hold as a vectoridentity once we strip away g. We have

$$\oint_C d\mathbf{r}'\,(\mathbf{r}\cdot\mathbf{r}') = \boldsymbol{\mathcal{S}}\times\mathbf{r}$$

where we've introduced the vector area S of the surface S bounded by C, defined as

$$\boldsymbol{\mathcal{S}} = \int_S d\mathbf{S}$$

If the boundary C lies in a plane – as it does for us – then the vector S points out of the plane.

1.8.2. General Current Distributions

We can now perform the same kind of expansion for a general current distribution J localised within some region of space. We use the Taylor expansion in the general form of the vector potential,

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0}{4\pi}\int d^3r'\,\left(\frac{J_i(\mathbf{r}')}{r} + \frac{J_i(\mathbf{r}')\,(\mathbf{r}\cdot\mathbf{r}')}{r^3} + \dots\right)$$

where we're using a combination of vector and index notation to help remember how the indices on the left and right-hand sides match up. The first term above vanishes. Heuristically, this is because currents can't stop and end, they have to go around in loops. This means that the contribution from one part must be cancelled by the current somewhere else. To see this mathematically, we use the slightly odd identity,

$$\partial_j(J_j r_i) = (\partial_j J_j)\,r_i + J_i = J_i$$

where the last equality follows from the continuity condition $\nabla \cdot J = 0$. Using this, we see that the first term in the equation of A_i is a total derivative (of $\partial/\partial r'_i$ rather than $\partial/\partial r_i$) which vanishes if we take the integral over R^3 and keep the current localized within some interior region.

$$\partial_j(J_j r_i r_k) = (\partial_j J_j) r_i r_k + J_i r_k + J_k r_i = J_i r_k + J_k r_i$$

Because J in equation of A_i is a function of r′, we actually need to apply this trick to the $J_i r'_j$ terms in the expression. We once again abandon the boundary term to infinity. Dropping the argument of J, we can use the identity above to write the relevant piece of the second term as,

$$\int d^3r'\ J_i r_j r'_j = \int d^3r'\ \frac{r_j}{2}(J_i r'_j - J_j r'_i) = \int d^3r'\ \frac{1}{2}(J_i(\mathbf{r}\cdot\mathbf{r}') - r'_i(\mathbf{J}\cdot\mathbf{r}))$$

But now this is in a form that is ripe for the vector product identity $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$. This means that we can rewrite this term as

$$\int d^3r'\ \mathbf{J}(\mathbf{r}\cdot\mathbf{r}') = \frac{1}{2}\mathbf{r} \times \int d^3r'\ \mathbf{J} \times \mathbf{r}'$$

With this in hand, we see that the long distance fall-off of any current distribution again takes the dipole form

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3}$$

now with the magnetic dipole moment given by the integral,

$$\mathbf{m} = \frac{1}{2}\int d^3r'\ \mathbf{r}' \times \mathbf{J}(\mathbf{r}')$$

1.9. Magnetic Forces

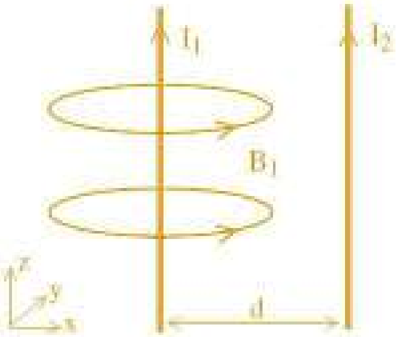
We've seen that a current produces a magnetic field. But a current is simply moving charge. And we know from the Lorentz force law that a charge q moving with velocity v will experience a force.

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

This means that if a second current is placed somewhere in the neighborhood of the first, then they will exert a force on one another. Our goal in this section is to figure out this force.

1.9.1. Force Between Currents

Let's start simple. Take two parallel wires carrying currents I_1 and I_2 respectively. We'll place them a distance d apart in the x direction. The current in the first wire sets up a magnetic field. So if the charges in the second wire are moving with velocity v, they will each experience a force



where ˆy is the direction of the magnetic field experienced by the second wire as shown in the Figure above. The next step is to write the velocity v in terms of the current I2 in the second wire. We did this when we first introduced the idea of currents: if there's a density n of these particles and each carries charge q, then the current density is

$$\mathbf{J}_2 = nq\mathbf{v}$$

For a wire with cross-sectional area A, the total current is just $I_2 = J_2A$. For our set-up, $J_2 = J_2$ˆz.Finally, we want to compute the force on the wire per unit length, f. Since the number of charges per unit length is nA and F is the force on each charge, we have,

$$\mathbf{f} = nA\mathbf{F} = \left(\frac{\mu_0 I_1 I_2}{2\pi d}\right)\hat{\mathbf{z}} \times \hat{\mathbf{y}} = -\left(\frac{\mu_0 I_1 I_2}{2\pi d}\right)\hat{\mathbf{x}}$$

This is our answer for the force between two parallel wires. If the two currents are in the same direction, so that I1I2 > 0, the overall minus sign means that the force between two wires is attractive. For currents in opposite directions, with I1I2 < 0, the force is repulsive.

1.9.2. Force and Energy for a Dipole

We start by asking a slightly different question. We'll forget about the second current and just focus on the first: call it J(r). We'll place this current distribution in a magnetic field B(r) and ask: what force does it feel? In general, there will be two kinds of forces. There will be a force on the centre of mass of the current distribution, which will make it move. There will also be a torque on the current distribution, which will want to make it re-orient itself with respect to the magnetic field. Here we're going to focus on the former. Rather remarkably, we'll see that we get the answer to the latter for free!

The Lorentz force experienced by the current distribution is

$$\mathbf{F} = \int_V d^3r\ \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})$$

We're going to assume that the current is localised in some small region r = R and that the magnetic field B varies only slowly in this region. This allows us to Taylor expand,

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{R}) + (\mathbf{r}\cdot\nabla)\mathbf{B}(\mathbf{R}) + \ldots$$

We then get the expression for the force

$$\mathbf{F} = -\mathbf{B}(\mathbf{R}) \times \int_V d^3r\ \mathbf{J}(\mathbf{r}) + \int_V d^3r\ \mathbf{J}(\mathbf{r}) \times [(\mathbf{r}\cdot\nabla)\mathbf{B}(\mathbf{R})] + \ldots$$

The first term vanishes because the currents have to go around in loops. We're going to do some fiddly manipulations with the second term. To help us remember that the derivative ∇ is acting on B, which is then evaluated at R, we'll introduce a dummy variable r′ and write the force as

$$\mathbf{F} = \int_V d^3r\ \mathbf{J}(\mathbf{r}) \times [(\mathbf{r}\cdot\nabla')\mathbf{B}(\mathbf{r}')]\Big|_{\mathbf{r}'=\mathbf{R}}$$

Now we want to play around with this. First, using the fact that ∇ × B = 0 in the vicinity of the second current, we're going to show, that we can rewrite the integrand as

$$\mathbf{J}(\mathbf{r}) \times [(\mathbf{r}\cdot\nabla')\mathbf{B}(\mathbf{r}')] = -\nabla' \times [(\mathbf{r}\cdot\mathbf{B}(\mathbf{r}'))\mathbf{J}(\mathbf{r})]$$

But the terms in the brackets are the components of ∇ × B and so vanish. So our result is true and we can rewrite the force,

$$\mathbf{F} = -\nabla' \times \int_V d^3r\ (\mathbf{r}\cdot\mathbf{B}(\mathbf{r}'))\,\mathbf{J}(\mathbf{r})\Big|_{\mathbf{r}'=\mathbf{R}}$$

Now we need to manipulate this a little more. We make use of the identity where we replace the constant vector by B. Thus, up to some relabelling, is the same as

$$\int_V d^3r\ (\mathbf{B}\cdot\mathbf{r})\mathbf{J} = \frac{1}{2}\mathbf{B} \times \int_V d^3r\ \mathbf{J}\times\mathbf{r} = -\mathbf{B}\times\mathbf{m}$$

where m is the magnetic dipole moment of the current distribution. Suddenly, our expression for the force is looking much nicer: it reads,

$$\mathbf{F} = \nabla \times (\mathbf{B} \times \mathbf{m})$$

where we've dropped the $r' = R$ notation because, having lost the integral, there's no cause for confusion: the magnetic dipole m is a constant, while B varies in space. Now we invoke a standard vector product identity. Using $\nabla \cdot B = 0$, this simplifies and we're left with a simple expression for the force on a dipole

$$F = \nabla(B \cdot m)$$

After all that work, we're left with something remarkably simple. Moreover, like many forces in Newtonian mechanics, it can be written as the gradient of a function. This function, of course, is the energy U of the dipole in the magnetic field,

$$U = -B \cdot m$$

This is an important expression that will play a role in later courses in Quantum Mechanics and Statistical Physics. For now, we'll just highlight something clever: we derived by considering the force on the centre of mass of the current. This is related to how U depends on r. But our final expression also tells us how the energy depends on the orientation of the dipole m at fixed position. This is related to the torque. Computing the force gives us the torque for free. This is because, ultimately, both quantities are derived from the underlying energy.

The Force Between Dipoles

As a particular example of the force, consider the case where the magnetic field is set up by a dipole m1. We know that the resulting long-distance magnetic field is,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_1}{r^3}\right)$$

Now we'll consider how this affects the second dipole $m = m_2$.

$$\mathbf{F} = \frac{\mu_0}{4\pi}\nabla\left(\frac{3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}}) - \mathbf{m}_1 \cdot \mathbf{m}_2}{r^3}\right)$$

where r is the vector from m1 to m2. Note that the structure of the force is identical to that between two electric dipoles. This is particularly pleasing because we used two rather different methods to calculate these forces. If we act with the derivative, we have

$$\mathbf{F} = \frac{3\mu_0}{4\pi r^4}\left[(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\mathbf{m}_2 + (\mathbf{m}_2 \cdot \hat{\mathbf{r}})\mathbf{m}_1 + (\mathbf{m}_1 \cdot \mathbf{m}_2)\hat{\mathbf{r}} - 5(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}\right]$$

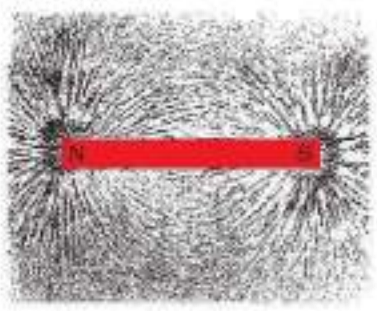
First note that if we swap m1 and m2, so that we also send $r \rightarrow -r$, then the force swaps sign. This is a manifestation of Newton's third law: every action has an equal and opposite reaction. Recall from Dynamics and Relativity lectures that we needed Newton's third law to prove the conservation of momentum of a collection of particles.

We see that this holds for a bunch of dipoles in a magnetic field. But there was also a second part to Newton's third law: to prove the conservation of angular momentum of a collection of particles, we needed the force to lie parallel to the separation of the two particles. And this is not true for the force. If you set up a collection of dipoles, they will start spinning, seemingly in contradiction of the conservation of angular momentum. What's going on?! Well, angular momentum is conserved, but you have to look elsewhere to see it. The angular momentum carried by the dipoles is compensated by the angular momentum carried by the magnetic field itself.

Finally, a few basic comments: the dipole force drops off as 1/r4, quicker than the Coulomb force. Correspondingly, it grows quicker than the Coulomb force at short distances. If m1 and m2 point in the same direction and lie parallel to the separation R, then the force is attractive. If m1 and m2 point in opposite directions and lie parallel to the separation between them, then the force is repulsive.

So What is a Magnet?

Until now, we've been talking about the magnetic field associated to electric currents. But when asked to envisage a magnet, most people would think if a piece of metal, possibly stuck to their fridge, possibly in the form of a bar magnet like the one shown in the picture. How are these related to our discussion above? These metals are permanent magnets. They often involve iron. They can be thought of as containing many microscopic magnetic dipoles, which align to form a large magnetic dipole M. In a bar magnet, the dipole M points between the two poles.



This means that the leading force between two magnets is described by our result. Suppose that M1, M2 and the separation R all lie along a line. If M1 and M2 point in the same direction, then the North pole of one magnet faces the South pole of another and tells us that the force is attractive. Alternatively, if M1 and M2 point in opposite directions then two poles of the same type face each other and the force is repulsive. This, of course, is what we all learned as kids. The only remaining question is: where do the microscopic dipole moments m come from? You might think that these are due to tiny electric atomic currents but this isn't quite right. Instead, they have a more fundamental origin. The electric charges — which are electrons — possess an inherent angular momentum called spin. Roughly you can think of the electron as spinning around its own axis in much the same way as the Earth spins. But, ultimately, spin is a quantum mechanical phenomenon and this classical analogy breaks down when pushed too far. The magnitude of the spin is:

$$s = \frac{1}{2}\hbar$$

where, recall, $\hbar$ has the same dimensions as angular momentum. We can push the classical analogy of spin just a little further. Classically, an electrically charged spinning ball would give rise to a magnetic dipole moment. So one may wonder if the spinning electron also gives rise to a magnetic dipole. The answer is yes. It is given by,

$$\mathbf{m} = g\frac{e}{2m}\mathbf{s}$$

where e is the charge of the electron and m is its mass. The number g is dimensionless and called, rather uninspiringly, the g-factor. It has been one of the most important numbers in the history of theoretical physics, with several Nobel prizes awarded to people for correctly calculating it! The classical picture of a spinning electron suggests g = 1. But this is wrong. The first correct prediction (and, correspondingly, first Nobel prize) was by Dirac. His famous relativistic equation for the electron gives,

$$g = 2$$

1.10. Units of Electromagnetism

More than any other subject, electromagnetism is awash with different units. In large part this is because electromagnetism has such diverse applications and everyone from astronomers, to electrical engineers, to particle physicists needs to use it. But it's still annoying. Here we explain the basics of SI units.

The SI unit of charge is the Coulomb. As of 2019, the Coulomb is defined in terms of the charge −e carried by the electron. This is taken to be exactly,

$$e = 1.602176634 \times 10^{-19}\ C$$

If you rub a balloon on your sweater, it picks up a charge of around 10^{-6} C or so. A bolt of lightening deposits a charge of about 15 C. The total charge that passes through an AA battery in its lifetime is about 5000 C. The SI unit of current is the Ampere, denoted A. It is defined as one Coulomb of charge passing every second. The current that runs through single ion channels in cell membranes is about 10^{-12} A. The current that powers your toaster is around 1 A to 10 A. There is a current in the Earth's atmosphere, known as the Birkeland current, which creates the aurora and varies between 105 A and 106 A. Galactic size currents in so-called Seyfert galaxies (particularly active galaxies) have been measured at a whopping 1018 A. The electric field is measured in units of NC^{-1}. The electrostatic potential ϕ has units of Volts, denoted V, where the 1 Volt is the potential difference between two infinite, parallel plates, separated by 1 m, which create an electric field of 1 NC^{-1}.

A nerve cell sits at around 10−2 V . An AA battery sits at 1.5 V . The largest manmade voltage is 107 V produced in a van der Graaf generator. This doesn’t compete well with what Nature is capable of. The potential difference between the ends of a lightening bolt can be 108 V . The voltage around a pulsar (a spinning neutron star) can be 1015 V .

The unit of a magnetic field is the Tesla, denoted T. A particle of charge 1 C, passing through a magnetic field of 1 T at 1 ms−1 will experience a force of 1 N. From the examples that we’ve seen above it’s clear that 1 C is a lot of charge. Correspondingly, 1 T is a big magnetic field. Our best instruments (SQUIDs) can detect changes in magnetic fields of 10−18 T. The magnetic field in your brain is 10−12 T. The strength of the Earth’s magnetic field is around 10−5 T while a magnet stuck to your fridge has about 10−3 T. The strongest magnetic field we can create on Earth is around 100 T.

Again, Nature beats us quite considerably. The magnetic field around neutron stars can be between 106 T and 109 T. (There is an exception here: in “heavy ion collisions”, in which gold or lead nuclei are smashed together in particle colliders, it is thought that magnetic fields comparable to those of neutron stars are created. However, these magnetic fields are fleeting and small. They are stretch over the size of a nucleus and last for a millionth of a second or so). As the above discussion amply demonstrates, SI units are based entirely on historical convention rather than any deep underlying physics. A much better choice is to pick units of charge such that we can discard ϵ0 and μ0. There are two commonly used frameworks that do this, called Lorentz-Heaviside units and Gaussian units. I should warn you that the Maxwell equations take a slightly different form in each. To fully embrace natural units, we should also set the speed of light c = 1. However we can’t set everything to one. There is one combination of the fundamental constants of Nature which is dimensionless. It is known as the fine structure constant,

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$$

and takes value α ≈ 1/137. Ultimately, this is the correct measure of the strength of the electromagnetic force. It tells us that, in units with $\epsilon_0 = \hbar = c = 1$, the natural, dimensionless value of the charge of the electron is e ≈ 0.3

A History of Magnetostatics

Magnetism is a science with more than two millennia of recorded history. The attraction of ferrous objects to a permanent magnet across a distance has been a source of curiosity since the iron age. Investigations of magnetic phenomena led to the invention of steel magnets — needles and horseshoes — and the compass that enabled the exploration of the planet. Feeble permanent magnets are widespread in Nature in the form of lodestones, pieces of rock that are mainly composed of the iron oxide mineral magnetite (Fe 3O4). Outcrops can be magnetized by the huge currents in lightning strikes and such natural magnets were known and studied in ancient Greece, Egypt, China and Mesoamerica. Aristotle attributed early thoughts on the nature of magnetic attraction to Thales, who was born in Milet in Asia Minor in 624 BC. He was an animist who credited the magnet with a soul on account of its ability to create movement. This strange idea was to persist in Europe until the 17th century. The magnet itself is thought to be named after Magnesia, a city in Asia Minor that was a source of lodestone. In the 5th century BC, when Empedokles postulated the existence of four elements, Earth, Water, Air and Fire; magnetism was related to Air with special effluvia somehow passing through pores in magnetic material, a theory echoed much later by Descartes.

A more productive approach was followed in China (Needham 1962), where magnetism was linked to geomancy. The art of adapting the residences of the living and the tombs of the dead to harmonize with local currents of the cosmic breath demanded a knowledge of its direction. A South-pointer consisting of a carved lodestone spoon free to rotate upon a polished baseplate (Fig.1) was in use in the first century BC, and may have been known hundreds of years earlier. An important discovery, attributed to Zheng Gongliang in 1064, was that iron could acquire a thermoremanent magnetization when rapidly cooled from red heat in the ambient magnetic field. A short step led to the suspended magnetic needle, which was described by Shen Kua around 1088 and seen to deviate, like the South-pointer, from a strictly North-South axis.

In the ensuing ferment in Europe and North America, it is often difficult to attribute credit for any advance, distinguishing between who had the idea, who devised a convincing experimental demonstration, and who created the first practical, working machine. Scientists and engineers belonged to different communities, with different approaches. 19th century electromagnetic science was driven, however, by Michael Faraday. Working entirely by observation and experimentation, with no dependence on formal theory, he developed the concept of ‘magnetic field’. Faraday classified materials as ferromagnetic (substances such as iron which were easily magnetized and drawn towards stronger magnetic fields), paramagnetic (substances which are attracted by the field but much more weekly than iron) or diamagnetic (materials which were pushed away by the field). Working with an electromagnet, he discovered the phenomenon of ‘electromagnetic induction’ ¾ that a flow of electricity can be induced by a changing magnetic field. He demonstrated electrical ↔ mechanical energy conversion in a model motor and a dynamo with steel permanent magnets, which were used in most of the early demonstrations, but electromagnets proved indispensable for later, practical electromagnetic machines. His conviction that a magnetic field should have an influence on light led to his 1845 discovery of the magneto-optic Faraday effect ¾ the rotation of the plane of polarization of light passing through a magnetic medium in a direction parallel to its magnetization.

A classical theory emerged to account simultaneously for magnetism and its relationship with electricity. Around 1824 Siméon Denis Poisson developed a mathematical theory of magnetostatics. Letters from Ampère to Fresnel and Faraday found after his death show that by 1822 he had considered the possibility that the currents causing ferromagnetism were ‘molecular’ rather than macroscopic in form. Wilhelm Weber formally presented the idea that ‘molecules’ of iron were capable of movement around their centres, suggesting that they lay in different directions in unmagnetized material, but aligned in a common direction in the presence of an applied magnetic field. This was the beginning of an explanation of hysteresis, the central phenomenon of technical magnetism, demonstrated experimentally by James Ewing using a board of small, pivoting magnets. He then built an ingenious instrument based on the bending of a ferromagnetic current-carrying wire in the gap of an electromagnet to measure B and H under AC and DC excitation.

The astonishing transformation of science and society that began in 1820 is known as the electromagnetic revolution. By the end of the century it was heralding the electrification of the planet and changing forever human communications, transportation and conditions of life and leisure. This was the second time that magnetic technology changed the world.

1.11. Electrodynamics

For static situations, Maxwell’s equations split into the equations of electrostatics, and the equations of magnetostatics. The only hint that there is a relationship between electric and magnetic fields comes from the fact that they are both sourced by charge: electric fields by stationary charge; magnetic fields by moving charge. In this section we will see that the connection becomes more direct when things change with time.

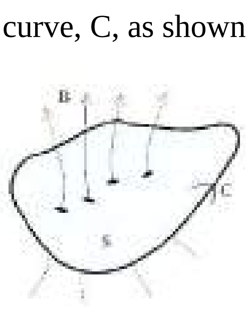
1.11.1. Faraday’s Law of Induction

“I was at first almost frightened when I saw such mathematical force made to bear upon the subject, and then wondered to see that the subject stood it so well.” Faraday to Maxwell, 1857

One of the Maxwell equations relates time varying magnetic fields to electric fields,

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

This equation tells us that if you change a magnetic field, you’ll create an electric field. In turn, this electric field can be used to accelerate charges which, in this context, is usually thought of as creating a current in wire. The process of creating a current through changing magnetic fields is called induction. We’ll consider a wire to be a conductor, stretched along a stationary, closed curve, C, as shown in the figure below.



We will refer to closed wires of this type as a “circuit”. We integrate both sides of over a surface S which is bounded by C,

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

By Stokes theorem, we can write this as,

$$\int_C \mathbf{E} \cdot d\mathbf{r} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

Recall that the line integral around C should be in the right-handed sense; if the fingers on your right-hand curl around C then your thumb points in the direction of dS. (This means that in the figure dS points in the same direction as B). To get the last equality above, we need to use the fact that neither C nor S change with time. Both sides of this equation are usually given names. The integral of the electric field around the curve C is called the electromotive force, E, or emf for short,

$$\mathcal{E} = \int_C \mathbf{E} \cdot d\mathbf{r}$$

It's not a great name because the electromotive force is not really a force. Instead it's the tangential component of the force per unit charge, integrated along the wire. Another way to think about it is as the work done on a unit charge moving around the curve C. If there is a non-zero emf present then the charges will be accelerated around the wire, giving rise to a current. The integral of the magnetic field over the surface S is called the magnetic flux Φ through S,

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

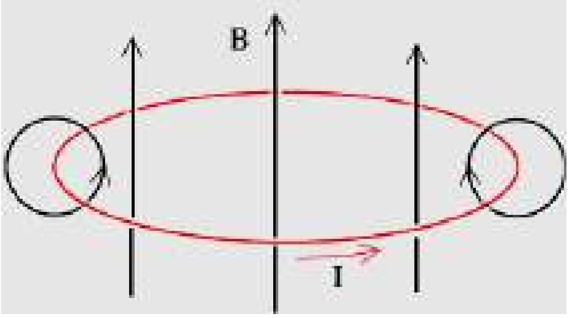
The Maxwell equation can be written as,

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

In this form, the equation is usually called Faraday's Law. Sometimes it is called the flux rule.

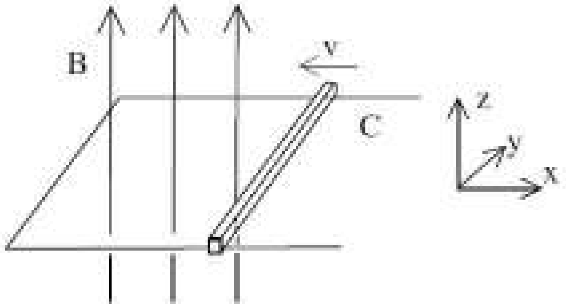
Faraday's law tells us that if you change the magnetic flux through S then a current will flow. There are a number of ways to change the magnetic field. You could simply move a bar magnet in the presence of circuit, passing it through the surface S; or you could replace the bar magnet with some other current density, restricted to a second wire C′, and move that; or you could keep the second wire C′ fixed and vary the current in it, perhaps turning it on and off. All of these will induce a current in C. However, there is then a secondary effect. When a current flows in C, it will create its own magnetic field. We've seen how this works for steady currents. This induced magnetic field will always be in the direction that opposes the change. This is called Lenz's law. If you like, "Lenz's law" is really just the minus sign in Faraday's law. We can illustrate this with a simple example. Consider the case where C is a circle, lying in a plane. We'll place it in a uniform B field and then make B smaller over time, so $\dot{\Phi} < 0$. By Faraday's law, $E > 0$ and the current will flow in the right-handed direction around C as shown. But now you can wrap your right-hand in a different way: point your thumb in the direction of the current and let your fingers curl to show you the direction of the induced magnetic field. These are the circles drawn in the figure. You see that the induced current causes B to increase inside the loop, counteracting the original decrease.

Lenz's law is rather like a law of inertia for magnetic fields. It is necessary that it works this way simply to ensure energy conservation: if the induced magnetic field aided the process, we'd get an unstable runaway situation in which both currents and magnetic fields were increasing forever.



1.11.2. Faraday's Law for Moving Wires:

There is another, related way to induce currents in the presence of a magnetic field: you can keep the field fixed, but move the wire. Perhaps the simplest example is shown in the figure: it's a rectangular circuit, but where one of the wires is a metal bar that can slide backwards and forwards. This whole set-up is then placed in a magnetic field, which passes up, perpendicular through the circuit.



Slide the bar to the left with speed v. Each charge q in the bar experiences a Lorentz force qvB, pushing it in the y direction. This results in an emf which, now, is defined as the integrated force per charge. In this case, the resulting emf is,

$$\mathcal{E} = vBd$$

where d is the length of the moving bar. But, because the area inside the circuit is getting smaller, the flux through C is also decreasing. In this case, it's simple to compute the change of flux: it is,

$$\frac{d\Phi}{dt} = -vBd$$

We see that once again the change of flux is related to the emf through the flux rule.

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

In this case, it is because the current involves charges moving with some speed u around the circuit. These too feel a Lorentz force law, now pushing the bar back to the right. This means that if you let the bar go, it will not continue with constant speed, even if the connection is frictionless. Instead it will slow down. This is the analog of Lenz's law in the present case.

1.12. Inductance and Magnetostatic Energy:

Suppose that a constant current I flows along some curve C. From the results of calculated before we know that this gives rise to a magnetic field and hence a flux $\Phi = \int B \cdot dS$ through the surface S bounded by C. Now increase the current I. This will increase the flux Φ. But we've just learned that the increase in flux will, in turn, induce an emf around the curve C. The minus sign of Lenz's law ensures that this acts to resist the change of current. The work needed to build up a current is what's needed to overcome this emf.

1.12.1. Inductance

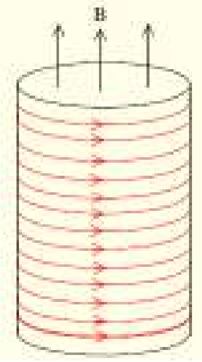
If a current I flowing around a curve C gives rise to a flux $\Phi = \int B \cdot dS$ then the inductance L of the circuit is defined to be,

$$L = \frac{\Phi}{I}$$

The inductance is a property only of our choice of curve C.

An Example: The Solenoid

A solenoid consists of a cylinder of length l and cross-sectional area A shown in figure below.



We take $l \gg \sqrt{A}$ so that any end-effects can be neglected. A wire wrapped around the cylinder carries current I and winds N times per unit length.

$$B = \mu_0 I N$$

This means that a flux through a single turn is $\Phi_0 = \mu_0 INA$. The solenoid consists of Nl turns of wire, so the total flux is,

$$\Phi = \mu_0 I N^2 A l = \mu_0 I N^2 V$$

with V = Al the volume inside the solenoid. The inductance of the solenoid is therefore

1.12.2. Magnetostatic Energy

The definition of inductance is useful to derive the energy stored in the magnetic field. Let's take our circuit C with current I. We'll try to increase the current. The induced emf is,

$$\mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$

As we mentioned above, the induced emf can be thought of as the work done in moving a unit charge around the circuit. But we have current I flowing which means that, in time δt, a charge Iδt moves around the circuit and the amount of work done is,

$$\delta W = \mathcal{E} I \delta t = -LI\frac{dI}{dt}\delta t \quad \Rightarrow \quad \frac{dW}{dt} = -LI\frac{dI}{dt} = -\frac{L}{2}\frac{dI^2}{dt}$$

The work needed to build up the current is just the opposite of this. Integrating over time, we learn that the total work necessary to build up a current I along a curve with inductance L is,

$$W = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi$$

Following our discussion for electric energy i, we identify this with the energy U stored in the system. We can write it as,

$$U = \frac{1}{2}I\int_S \mathbf{B}\cdot d\mathbf{S} = \frac{1}{2}I\int_S \nabla\times\mathbf{A}\cdot d\mathbf{S} = \frac{1}{2}I\oint_C \mathbf{A}\cdot d\mathbf{r} = \frac{1}{2}\int d^3x\ \mathbf{J}\cdot\mathbf{A}$$

where, in the last step, we've used the fact that the current density J is localised on the curve C to turn the integral into one over all of space. At this point we turn to the Maxwell equation $\nabla \times B = \mu_0 J$ to write the energy as,

$$U = \frac{1}{2\mu_0}\int d^3x\ (\nabla\times\mathbf{B})\cdot\mathbf{A} = \frac{1}{2\mu_0}\int d^3x\ \left[\nabla\cdot(\mathbf{B}\times\mathbf{A}) + \mathbf{B}\cdot(\nabla\times\mathbf{A})\right]$$

We assume that B and A fall off fast enough at infinity so that the first term vanishes. We're left with the simple expression,

$$U = \frac{1}{2\mu_0}\int d^3x\ \mathbf{B}\cdot\mathbf{B}$$

Combining this with the equation of the electric field, we have the energy
stored in the electric and magnetic fields,

$$U = \int d^3x \left(\frac{\epsilon_0}{2} \mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0} \mathbf{B} \cdot \mathbf{B} \right)$$

We had to approach the energy in both the electric and magnetic fields in a rather indirect manner, by focussing not on the fields but on the work done to assemble the necessary charges and currents. There's nothing wrong with this, but it's not a very elegant approach and it would be nice to understand the energy directly from the fields themselves. Second, we computed the energy for the electric fields and magnetic fields alone and then simply added them. We can't be sure, at this point, that there isn't some mixed contribution to the energy such as E.B.

1.13. Resistance:

You may have noticed that our discussion above has been a little qualitative. If the flux changes, we have given expressions for the induced emf E but we have not given an explicit expression for the resulting current. And there's a good reason for this: it's complicated. The presence of an emf means that there is a force on the charges in the wire. And we know from Newtonian mechanics that a force will cause the charges to accelerate. This is where things start to get complicated. Accelerating charges will emit waves of electromagnetic radiation, a process that you will explore later. Relatedly, there will be an opposition to the formation of the current through the process that we've called Lenz's law.

So things are tricky. What's more, in real wires and materials there is yet another complication: friction. Throughout these lectures we have modelled our charges as if they are moving unimpeded, whether through the vacuum of space or through a conductor. But that's not the case when electrons move in real materials. Instead, there's stuff that gets in their way: various messy impurities in the material, or sound waves (usually called phonons in this context) which knock them off-course, or even other electrons. All these effects contribute to a friction force that acts on the moving electrons. The upshot of this is that the electrons do not accelerate forever. In fact, they do not accelerate for very long at all. Instead, they very quickly reach an equilibrium speed, analogous to the "terminal velocity" that particles reach when falling in gravitational field while experiencing air resistance. In many circumstances, the resulting current I is proportional to the applied emf. This relationship is called Ohm's law. It is,

$$\mathcal{E} = IR$$

The constant of proportionality R is called the resistance. The emf is $\varepsilon = \int E.dx$. If we write E = −∇ϕ, then E = V , the potential difference between two ends of the wire. This gives us the version of Ohm's law that is familiar from school: V = IR. The resistance R depends on the size and shape of the wire. If the wire has length L and cross-sectional area A, we define the resistivity as ρ = AR/L. Alternatively, we talk about the conductivity σ = 1/ρ.

The general form of Ohm's law is then,

$$\mathbf{J} = \sigma \mathbf{E}$$

Unlike the Maxwell equations, Ohm’s law does not represent a fundamental law of Nature. It is true in many, perhaps most, materials. But not all. There is a very simple classical model, known as the Drude model, which treats electrons as billiard balls experiencing linear drag which gives rise to Ohm’s law.. But a proper derivation of Ohm’s law needs quantum mechanics and a more microscopic understanding of what’s happening in materials. Needless to say, this is (way) beyond the scope of this course. So, at least in this small section, we will take Ohm’s law as an extra input in our theory.

When Ohm’s law holds, the physics is very different. Now the applied force (or, in this case, the emf) is proportional to the velocity of the particles rather than the acceleration. It’s like living in the world that Aristotle envisaged rather than the one Galileo understood. But it also means that the resulting calculations typically become much simpler.

- **An Example**

Let’s return to our previous example of a slidying bar of length d and mass m which forms a circuit, sitting in a magnetic field $B = B\hat{z}$. But now we will take into account the effect of electrical resistance. We take the resistance of the sliding bar to be R. But we’ll make life easy for ourselves and assume that the resistance of the rest of the circuit is negligible.

There are two dynamical degrees of freedom in our problem: the position x of the sliding bar and the current I that flows around the circuit. We take I > 0 if the current flows along the bar in the positive $\hat{y}$ direction. The Lorentz force law tells us that the force on a small volume of the bar is $F = IB\widehat{y\hat{z}}$. The force on the whole bar is therefore,

$$\mathbf{F} = IBd\,\hat{\mathbf{x}}$$

The equation of motion for the position of the wire is then,

$$m\ddot{x} = IBd$$

Now we need an equation that governs the current I(t). If the total emf around the circuit comes from the induced emf, we have,

$$\mathcal{E} = -\frac{d\Phi}{dt} = -Bd\dot{x}$$

Ohm’s law tells us that E = IR. Combining these, we get a simple differential equation for the position of the bar,

$$m\ddot{x} = -\frac{B^2d^2}{R}\,\dot{x}$$

which we can solve to see that any initial velocity of the bar, v, decays exponentially:

$$\dot{x}(t) = -ve^{-B^2d^2t/mR}$$

Note that, in this calculation we neglected the magnetic field created by the current. It's simple to see the qualitative effect of this. If the bar moves to the left, so $\dot{x} < 0$, then the flux through the circuit decreases. The induced current is I > 0 which increases B inside the circuit which, in accord with Lenz's law, attempts to counteract the reduced flux.

In the above derivation, we assumed that the total emf around the circuit was provided by the induced emf. This is tantamount to saying that no current flows when the bar is stationary. But we can also relax this assumption and include in our analysis an emf E_0 across the circuit (provided, for example, by a battery) which induces a current $I_0 = E_0d/R$. Now the total emf is,

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_{\text{induced}} = \mathcal{E}_0 - Bd\dot{x}$$

The total current is again given by Ohms law I = E/R. The position of the bar is now governed by the equation,

$$m\ddot{x} = -\frac{Bd}{R}\left(\mathcal{E}_0 - Bd\dot{x}\right)$$

1.14. The Displacement Current

We've now worked our way through most of the Maxwell equations. We've looked at Gauss' law (which is really equivalent to Coulomb's law)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

and the law that says there are no magnetic monopoles,

$$\nabla \cdot \mathbf{B} = 0$$

and Amp`ere's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

and now also Faraday's law

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

In fact, there's only one term left to discuss. When fields change with time, there is an extra term that appears in Amp`ere's law, which reads in full:

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

This extra term is called the displacement current. It's not a great name because it's not a current. Nonetheless, as you can see, it sits in the equation in the same place as the current which is where the name comes from. The displacement current term is different. This was arrived at by pure thought alone. This is one of Maxwell's contributions to the subject and, in part, why his name now lords over all four equations.

1.14.1. Adding Ampere's Law with Displacement Current:

Let's now see how adding the displacement current fixes the situation. We'll first look at the abstract issue that Ampere's law requires ∇.J = 0. If we add the displacement current, then taking the divergence of gives,

$$\mu_0\left(\nabla\cdot\mathbf{J}+\epsilon_0\nabla\cdot\frac{\partial\mathbf{E}}{\partial t}\right)=\nabla\cdot(\nabla\times\mathbf{B})=0$$

But, using Gauss's law, we can write $\epsilon_0\nabla.E = \rho$, so the equation above becomes

$$\nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}=0$$

which is the continuity equation that tells us that electric charge is locally conserved. It's only with the addition of the displacement current that Maxwell's equations become consistent with the conservation of charge.

Now let's return to our puzzle of the circuit and capacitor. Without the displacement current we found that B = 0 when we chose the surface S′ which passes between the capacitor plates. But the displacement current tells us that we missed something, because the build up of charge on the capacitor plates leads to a time-dependent electric field between the plates.

$$E=\frac{Q}{\epsilon_0 A}$$

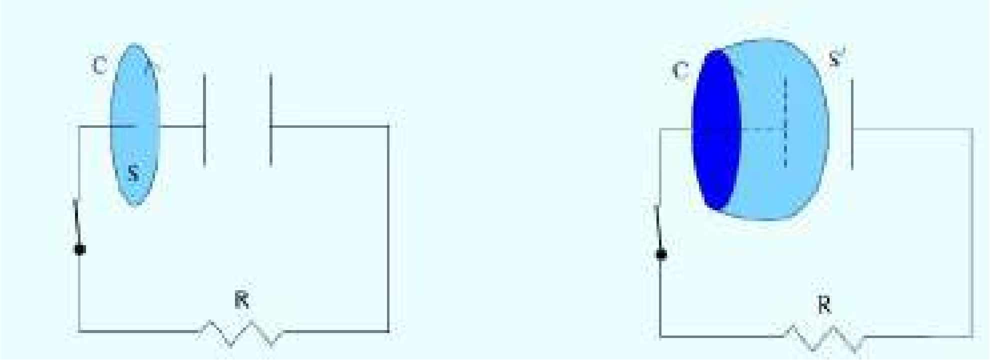
where A is the area of each plate and Q is the charge that sits on each plate, and we are ignoring the edge effects which is acceptable as long as the size of the plates is much bigger than the gap between them. Since Q is increasing over time, the electric field is also increasing,

$$\frac{\partial E}{\partial t}=\frac{1}{\epsilon_0 A}\frac{dQ}{dt}=\frac{1}{\epsilon_0 A}I(t)$$

By repeating the calculation ,we get an extra term,

$$\int_C \mathbf{B}\cdot d\mathbf{r}=\int_{S'}\mu_0\epsilon_0\frac{\partial E}{\partial t}=\mu_0 I$$

From this equation, we found using Ampere's law applied to the surface S.The figure below contradicts with the ampere's law and displacement current,



1.14.2. Light waves in terms of electromagnetic Maxwell's equation:

We've derived two wave equations, one for E and one for B. We can solve these independently, but it's important to keep in our mind that the solutions must also obey the original Maxwell equations. This will then give rise to a relationship between E and B. Let's see how this works. We'll start by looking for a special class of solutions in which waves propagate in the x-direction and do not depend on y and z. These are called plane-waves because, by construction, the fields E and B will be constant in the (y, z) plane for fixed x and t. The Maxwell equation $\nabla \cdot E = 0$ tells us that we must have E_x constant in this case. Any constant electric field can always be added as a solution to the Maxwell equations so, without loss of generality, we'll choose this constant to vanish. We look for solutions of the form

$$\mathbf{E} = (0, E(x,t), 0)$$

where E satisfies the wave equation which is now,

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \nabla^2 E = 0$$

The most general solution to the wave equation takes the form

$$E(x,t) = f(x-ct) + g(x+ct)$$

Here f(x−ct) describes a wave profile which moves to the right with speed c. (Because, as t increases, x also has to increase to keep f constant). Meanwhile, g(x+ct) describes a wave profile moving to the left with the speed c.

The most important class of solutions of this kind are those which oscillate with a single frequency ω. Such waves are called monochromatic. For now, we'll focus on the right-moving waves and take the profile to be the sine function. We have,

$$E = E_0 \sin\left[\omega\left(\frac{x}{c} - t\right)\right]$$

We usually write this as,

$$E = E_0 \sin(kx - \omega t)$$

Where k is the wave number,

Equations of this kind, expressing frequency in terms of wavenumber, are called dispersion relations. Because waves are so important in physics, there's a whole bunch of associated quantities which we can define. They are:

- The quantity ω is more properly called the angular frequency and is taken to be positive. The actual frequency f = ω/2π measures how often a wave peak passes you by. But because we will only talk about ω, we will be lazy and just refer to this as frequency.
- The period of oscillation is T = 2π/ω.
- The wavelength of the wave is λ = 2π/k. This is the property of waves that

you first learn about in kindergarten. The wavelength of visible light is between λ ~ 3.9×10^{-7} m and 7×10^{-7} m. Although we grow up thinking about wavelength, moving forward the wavenumber
k will turn out to be a more useful description of the wave.

- E_0 is the amplitude of the wave.

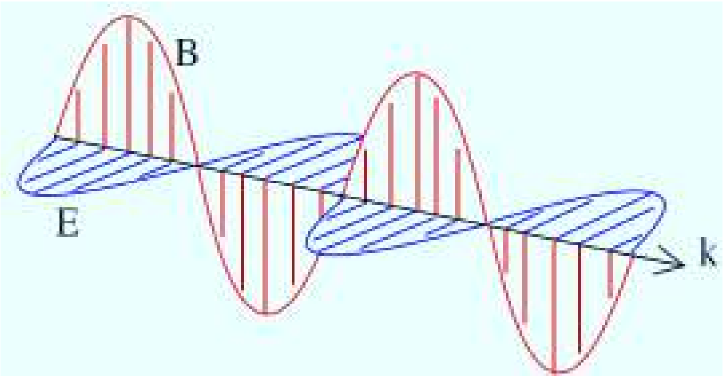
So far we have only solved for the electric field. To determine the magnetic field, we use $\nabla \cdot B = 0$ to tell us that B_x is constant and we again set $B_x = 0$. We know that the other components B_y and B_z must obey the wave equation. But their behaviour is dictated by what the electric field is doing through the Maxwell equation $\nabla \times E = -\partial B/\partial t$. This tells us that,

$$\mathbf{B} = (0, 0, B)$$

$$\frac{\partial B}{\partial t} = -\frac{\partial E}{\partial x} = -kE_0 \cos(kx - \omega t)$$

$$B = \frac{E_0}{c} \sin(kx - \omega t)$$

We see that the electric E and magnetic B fields oscillate in phase, but in orthogonal directions. And both oscillate in directions which are orthogonal to the direction in which the wave travels



Because the Maxwell equations are linear, we're allowed to add any number of solutions. This sometimes goes by the name of the principle of superposition. This is a particularly important property in the context of light, because it's what allow light rays travelling in different directions to pass through each other. In other words, it's why we can see anything at all.

The linearity of the Maxwell equations also encourages us to introduce some new notation which, at first sight, looks rather strange.

$$\mathbf{E} = E_0 \hat{\mathbf{y}} \, e^{i(kx-\omega t)} \quad , \quad \mathbf{B} = \frac{E_0}{c} \hat{\mathbf{z}} \, e^{i(kx-\omega t)}$$

This is strange because the physical electric and magnetic fields should certainly be real objects. You should think of them as simply the real parts of the expressions above. But the linearity of the Maxwell equations means both real and imaginary parts of E and B solve the Maxwell equations. And, more importantly, if we start adding complex E and B solutions, then the resulting real and imaginary pieces will also solve the Maxwell equations. The advantage of this notation is simply that it's typically easier to manipulate complex numbers than lots of cos and sin formulae.

However, you should be aware that this notation comes with some danger: whenever you compute something which isn't linear in E and B — for example, the energy stored in the fields, which is a quadratic quantity — you can't use the complex notation above; you need to take the real part first.

1.14.3. Polarisation:

Above we have presented a particular solution to the wave equation. Let's now look at the most general solution with a fixed frequency ω. This means that we look for solutions within the ansatz,

$$\mathbf{E} = \mathbf{E}_0 \, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \quad \text{and} \quad \mathbf{B} = \mathbf{B}_0 \, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

where, for now, both E0 and B0 could be complex-valued vectors. (Again, we only get the physical electric and magnetic fields by taking the real part of these equations). The vector k is called the wavevector. Its magnitude, |k| = k, is the wavenumber and the direction of k points in the direction of propagation of the wave.

We get further constraints on E_0, B_0 and k from the original Maxwell equations. These are,

$$\begin{aligned} \nabla \cdot \mathbf{E} = 0 \quad &\Rightarrow \quad i\mathbf{k} \cdot \mathbf{E}_0 = 0 \\ \nabla \cdot \mathbf{B} = 0 \quad &\Rightarrow \quad i\mathbf{k} \cdot \mathbf{B}_0 = 0 \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad &\Rightarrow \quad i\mathbf{k} \times \mathbf{E}_0 = i\omega \mathbf{B}_0 \end{aligned}$$

1.14.3.1. Linear Polarisation

Suppose that we take E_0 and B_0 to be real. The first two equations above say that both E_0 and B_0 are orthogonal to the direction of propagation. The last of the equations above says that E_0 and B_0 are also orthogonal to each other. You can check that the fourth Maxwell equation doesn't lead to any further constraints. Using the dispersion relation ω = ck, the last constraint above can be written as,

$$\hat{\mathbf{k}} \times (\mathbf{E}_0/c) = \mathbf{B}_0$$

This means that the three vectors ˆk, E_0/c and B_0 form a right-handed orthogonal triad. Waves of this form are said to be linearly polarised. The electric and magnetic fields oscillate in fixed directions, both of which are transverse to the direction of propagation,

1.14.3.2. Circular and Elliptic Polarisation

Suppose that we now take E_0 and B_0 to be complex. The actual electric and magnetic fields are consist of the real parts, but now the polarisation does not point in a fixed direction. To see this, write,

$$\mathbf{E}_0 = \boldsymbol{\alpha} - i\boldsymbol{\beta}$$

The real part of the electric field is then

$$\mathbf{E} = \boldsymbol{\alpha}\cos(\mathbf{k}\cdot\mathbf{x} - \omega t) + \boldsymbol{\beta}\sin(\mathbf{k}\cdot\mathbf{x} - \omega t)$$

with Maxwell equations ensuring that α ・ k = β ・ k = 0. If we look at the direction of E at some fixed point in space, say the origin x = 0, we see that it doesn't point in a fixed direction. Instead, it rotates over time within the plane spanned by α and β (which is the plane perpendicular to k).

A special case arises when the phase of E_0 is $e^{i\pi/4}$, so that |α| = |β|, with the further restriction that α ・ β = 0. Then the direction of E traces out a circle over time in the plane perpendicular to k. This is called circular polarisation. The polarisation is said to be right-handed if β = ˆk × α and left-handed if β = −ˆk .

Possible numerical problems:

1. Calculate the electric potential at a distance of 0.2 meters from a point charge of 3μC.
2. Calculate the energy stored in a capacitor with a capacitance of 10μF when it is charged to a potential difference of 50V.
3. A conductor of length 0.5 meters moves at a speed of 2m/s perpendicular to a magnetic field of strength 0.1T. Calculate the induced EMF in the conductor.
4. Calculate the force on a 0.3 meter length of wire carrying a current of 5A in a magnetic field of 0.2T. The wire is perpendicular to the magnetic field.
5. In an RC circuit, a 100 μF capacitor is initially charged to 12 V and then allowed to discharge through a 200 Ω resistor. Calculate the voltage across the capacitor after 5 seconds.
6. Consider a parallel plate capacitor whose plates are closely spaced. Let R be the radius of the plates and the current in the wire connected to the plates is 5 A, calculate the displacement current through the surface passing between the plates by directly calculating the rate of change of flux of electric field through the surface.
7. A transmitter consists of LC circuit with an inductance of 1 μH and a capacitance of 1 μF. What is the wavelength of the electromagnetic waves it emits?
8. Let an electromagnetic wave propagate along the x direction, the magnetic field oscillates at a frequency of 1010 Hz and has an amplitude of 10−5T, acting along the y - direction. Then, compute the wavelength of the wave. Also write down the expression for electric field in this case.

9. If the relative permeability and relative permittivity of the medium is 1.0 and 2.25, respectively. Find the speed of the electromagnetic wave in this medium.
10. The relative magnetic permeability of the medium is 2.5 and the relative electrical permittivity of the medium is 2.25. Compute the refractive index of the medium.
11. Compute the speed of the electromagnetic wave in a medium if the amplitude of electric and magnetic fields are 3 × 104 N C-1 and 2 × 10-4 T, respectively.
12. Calculate the magnetic field inside a solenoid with 500 turns, a length of 0.2 meters, and a current of 2 amperes flowing through it.
13. Calculate the force on a charge of 3μC moving with a velocity of 2×10^5 m/s, perpendicular to a magnetic field of 0.1T.
14. A rectangular loop with dimensions 0.2 meters by 0.3 meters rotates at 50 revolutions per second in a uniform magnetic field of 0.4T. Calculate the maximum induced EMF in the loop.
15. Calculate the force per unit length between two parallel wires 0.1 meters apart, each carrying a current of 5 amperes in the same direction.

CHAPTER 2:

DIELECTRICS AND MAGNETIC PROPERTIES OF MATERIAL

Chapter 2

2.1. Introduction

Dielectrics are insulating or non-conducting ceramic materials and are used in many applications such as capacitors, memories, sensors and actuators. Dielectrics are insulating materials that exhibit the property of electrical polarization, thereby they modify the dielectric function of the vacuum. A dielectric material is any material that supports charge without conducting it to a significant degree. In principle all insulators are dielectric, although thecapacity to support charge varies greatly between different insulators. Although these materials do not conduct electrical current when an electric field is applied, they are not inert to the electricfield. The field may cause a slight shift in the balance of charge within the material to form an electric dipole. Thus the materials is called dielectric material.

Dielectric materials are used in many applications, from simple electrical insulation to sensors and circuit components.

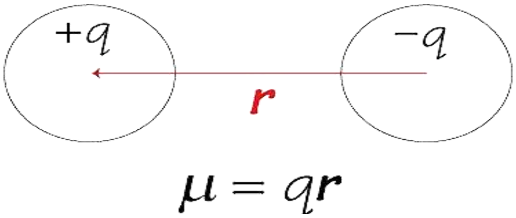
Faraday was carried out the first numerical measurements on the properties of insulating materials when placed between the two parallel plates (capacitor), those materials, he called as dielectrics. He has found that the capacity of a condenser was dependent on the nature of the material separating the conducting surface. This discovery encouraged further empirical studies of insulating materials aiming at maximizing the amount of charge that can be stored by a capacitor. In search of suitable dielectric materials for specific applications, these materials have become increasingly concerned with the detailed physical mechanism governing the behavior of these materials.

The difference between dielectric material and insulator depends on its application.Insulating materials are used to resist flow of current through it, on the other hand dielectric materials are used to store electrical energy. In contrast to the insulation aspect, the dielectric phenomena have become more general and fundamental, as it has the origin with the dielectric polarization.

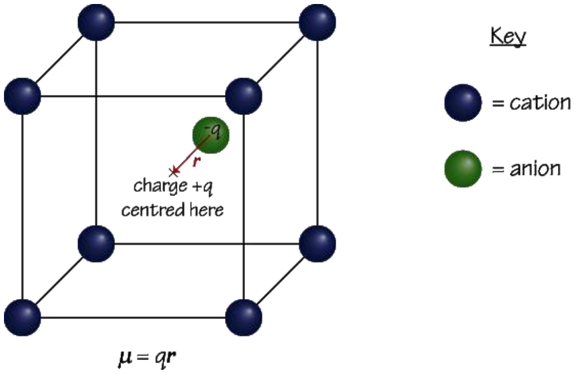
2.2. Electric dipoles:

Upon application of a dc or static electric field, there is a long range migration of charges.However, there is a limited movement of charges leading to the formation of charge dipoles and the material, in this state, is considered as polarized. These dipoles are aligned in the direction of the applied field. The net effect is called Polarization of the material. A dielectric supports charge by acquiring a polarisation in an electric field, whereby one surface develops a net positive charge while the opposite surface develops a net negative charge. This is made possible by the presence of electric dipoles – two opposite charges separated by a certain distance – on a microscopic scale.

1. If two discrete charged particles of opposite charges are separated by a certain distance, adipole moment **μ** arises.



2. If the centre of positive charge within a given region and the centre of negative charge within the same region are not in the same position, a dipole moment **μ** arises. For example, in the diagram below the centre of positive charge from the 8 cations shown is at X, while the centre ofnegative charge is located some distance away on the anion.



The second view of dipole moment is more useful, since it can be applied over a large areacontaining many charges in order to find the net dipole moment of the material. The dipoles can be aligned as well as be induced by the applied field.

Note that in the equation for dipole moment, **r** is a vector (the sign convention is that **r** pointsfrom negative to positive charge) therefore the dipole moment **μ** is also a vector

2.3. Electric field intensity or electric field strength (E)

The force experienced by a unit test charge is known as electric field strength E

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

where ε_0 is the permittivity or dielectric constant of the medium in which electric charge is placed.

For vacuum ε_0 = = 8.854 X10^{-12}Fm^{-1}

2.4. Electric flux density or electric displacement vector (D)

The electric flux density or electric displacement vector is the number of flux lines crossing normal to a unit surface area. The electric flux density at a distance from the point charge Q is

$$D = \frac{Q}{4\pi r^2}$$

then from above two equations D =εE

2.5. Dielectric constant (ε_r)

The dielectric constant of a material is defined as the ratio of the permittivity of the medium (ε) to the permittivity of free space (ε_o). It can also defined as the ratio of the capacitance with dielectric (C_d) and with air (C_A) between the plates.

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_o} = \frac{C_d}{C_A}$$

2.6. Capacitance:

The property of a conductor or system of conductor that describes its abilityto store electric charge.

C = q / V = A ε / d where

C is capacitance of capacitor

q is charge on the capacitor plate

V is potential difference between plates

A is area of capacitor plate

ε is permittivity of medium,

d is distance between capacitor plates

Units: Farad .

2.7. Polarization

When an electric field is applied to a material with dielectrics, the positive charges are displaced opposite to the direction of the field and negative charges displaced in the direction of the field. The displacement of these two charges create a local dipole, creation of dipole by applying electric field is called as polarization Polarization is defined as induced dipole moment per unit volume.

$$P = \frac{\mu}{volume}$$

2.8. Polarizability

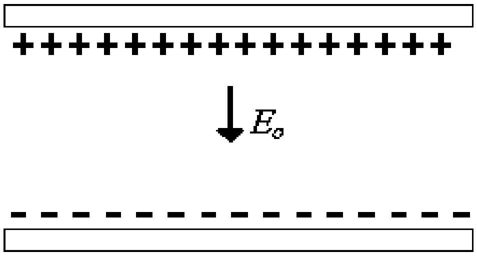
The polarization P is directly proportional to the electric field strength E

$$P \propto E$$
$$P = \alpha E$$

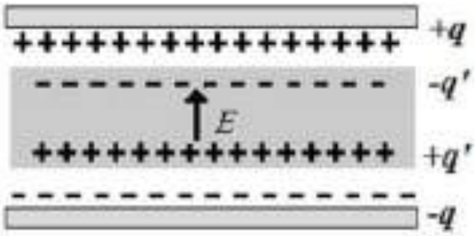
Where α proportionality is constant called as polarisability. The polarisability is defined as polarization per unit applied electric field. If the material contains N number of dipoles per unit volume then

$$P = N\alpha$$

2.8.1. Relation between polarization and dielectric constant



Parallel plate condensor without dielectrics



Parallel plate condensor with dielectrics

Let us apply Gauss theorem for parallel plate condenser.

$$\int E_0 dA = \frac{q}{\varepsilon_0}$$

$$E_0.A = \frac{q}{\varepsilon_0}$$

$$E_0 = \frac{q}{A\varepsilon_0} = \frac{\sigma}{\varepsilon_0}.$$

Let a dielectric slab placed between two plates. Due to polarization, charges appear on the two faces of the slab, and establish yet another field within the dielectric media. Let this field be E'. The direction of E' will be opposite to that of E_0.

The resultant field E in the material can be written as,

$$E = E_0 - E'$$

If σ_p is the charge/unit area on the inserted dielectric slab surfaces, then by following first equation, we write,

$$E' = \frac{q'}{A\varepsilon_0} = \frac{\sigma_p}{\varepsilon_0}.$$

$$E = \frac{\sigma}{\varepsilon_0} - \frac{\sigma_p}{\varepsilon_0}.$$

$$\varepsilon_0 E = \sigma - \sigma_p$$

Since the magnitude of polarization P = dipole moment/ Unit Volume

But dipole moment = induced charge X distance

Therefore

$$P = \text{induced charge/ Area} = \sigma_p$$

We know that electric displacement

field or electric flux density D is given by charge /unit areaD = q/A= σ

Therefore, it becomes

$$\varepsilon_0 E = D - P$$

$$P = D - E\varepsilon_0$$

in free space where there is no dielectric P=0

$$D = E\varepsilon_0$$

But in dielectric media the D changes. From electrostatics

$$D = \varepsilon_0 \varepsilon_r E$$

From equation of P,

$$P = E\varepsilon_o\varepsilon_r - E\varepsilon_0$$

$$(\varepsilon_r - 1) = \frac{P}{E\varepsilon_0} = \chi$$

Where χ is electric susceptibility of the dielectric medium. It doesn't have any units.Since P and E are vectors can be written as,

$$\vec{P} = \vec{E}\varepsilon_0(\varepsilon_r - 1)$$

This equation represents polarization vector.

2.8.2. Types of polarization

Dielectric polarization is the displacement of charge particles with the applied electric field. The displacement of electric charges results in formation of electric dipole moment in atoms, ions or molecules of the material. There are four different types of polarization, they are listed below.

a. Electric polarization,
b. Ionic polarization,
c. Orientation polarization
d. Space charge polarization

2.8.2.1. Electric polarization

The displacement of the positively charged nucleus and the negatively charged electrons of an atom in opposite directions, on application of an electric field, result in electronic polarization.

On applying a field, the electron cloud around the nucleus shifts towards the positive end of the field. As the nucleus and electron cloud are separated by a distance, dipole moment is created within each atom. The extent of this shift is proportional to the field strength.

Induced dipole moment

$$\mu_e \propto E$$

$$\mu_e = \alpha_e E$$

Where α_e is called electronic polarizability. The dipole moment per unit volume is called electronic polarization.

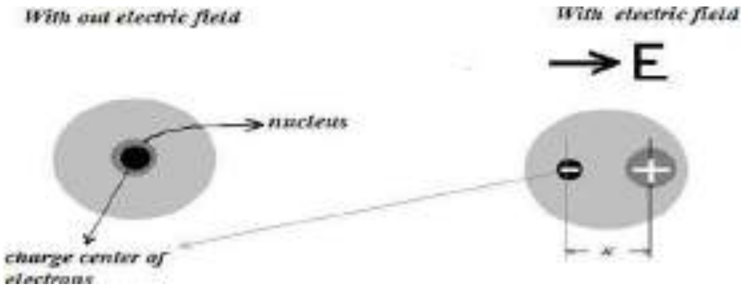
2.8.2.2. Calculation of electronic polarizability:

Electronic polarization can be explained by classical model of an atom in gasses. In gases the atoms are assumed that the interaction among the atoms is negligible. Here the nucleus of charge Ze is surrounded by an electron cloud of charge –Ze distributed in the sphere of radius R.

Charge density, $\rho = -\frac{Ze}{(4/3)\pi R^3}$

When an electric field E is applied, the nucleus and electrons experience Lorentz force of magnitude ZeE in opposite direction. Therefore the nucleus and electrons are pulled apart. As they are pulled apart a Coulomb force develops between them. At equilibrium these two forces are equal and nucleus and electron cloud are separated by a small distance x.

- It increases with increase of volume of the atom.
- This kind of polarization is mostly exhibited in monoatomic gases.(e.g. He, Ne, Ar, Kr, Xe etc..)
- It is independent of temperature.
- It occurs only at optical frequencies (10^{15}Hz)
- Vast fast process: 10^{-15}~10^{-16}s.



Lorentz force = $-ZeE$

Coulomb Force = Ze X $\dfrac{Ch\arg eenclosed\ \text{int}\ hesphereofradiusx}{4\pi\varepsilon_0 x^2}$

The charge enclosed = $\dfrac{4}{3}\pi x^3 \rho$

$$= \quad \frac{4}{3}\pi x^3\left[-\frac{3}{4}\left(\frac{Ze}{\pi R^3}\right)\right]$$

$$= \quad -\frac{Zex^3}{R^3}$$

Hence Coulomb force is = $\dfrac{Ze}{4\pi\varepsilon_0 x^2}$ X $-\dfrac{Zex^3}{R^3} = -\dfrac{Z^2e^2x}{4\pi\varepsilon_0 R^3}$

At equilibrium Lorentz force = Coulomb force

$$-ZeE = -\frac{Z^2e^2x}{4\pi\varepsilon_0 R^3}$$

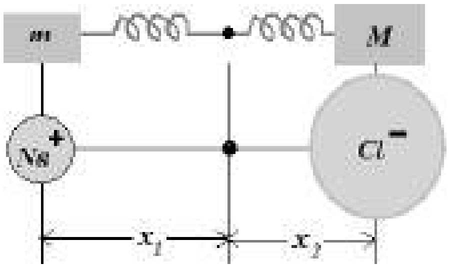
$$x = -\frac{4\pi\varepsilon_0 R^3 E}{Ze}$$

The displacement of the electron cloud is proportional to applied electric field

2.8.2.3. Ionic Polarization

Ionic polarization occurs in ionic solids such as NaCl, KBr, and LiBr. When an electric field is applied to an ionic solid the positive and negative ions displace to their respectivepolarities creating an electric dipole this is called as ionic polarization.

In the absence of an electric field there is no displacement of ions. When an electric field is applied an induceddipole moment μ_i is produced. Let x_1 and x_2 be the displacement ofpositive and negative ion respectively. Then the induced dipole moment.



$$\mu_i = e(x_1 + x_2)$$

Let F be restoring force F

$$\propto x_1$$
$$\propto x_2$$
$$= \beta_1 x_1$$
$$= \beta_2 x_2$$

From mechanics the spring constant of mass attached to a spring is given by eE =mω^2 x

At equilibrium the Lorentz force = restoring force

$$eE = m\omega_0^2 x$$

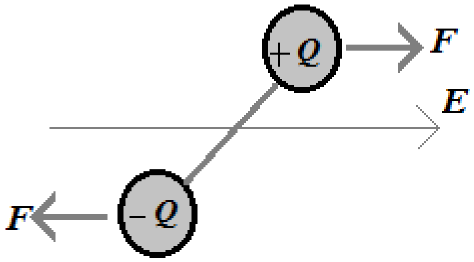
Therefore $x = \frac{eE}{m\omega_0^2}$ Then $x_1 = \frac{eE}{m\omega_0^2}$ $x_2 = \frac{eE}{M\omega_0^2}$

$$\mu = \frac{e^2}{\omega_0^2}\left[\frac{1}{m} + \frac{1}{M}\right]E = \alpha_i E$$

Where $\alpha_i = \frac{e^2}{\omega_0^2}\left[\frac{1}{m} + \frac{1}{M}\right]$ is called as ionic polarisability

2.8.2.4. Orientation Polarization

Orientation polarization occurs only in polar molecules (the molecules which have permanent dipole moment eg H_2O, Phenol, etc.). When an electric field is applied to a polar molecule, the dipoles experience a torque and try to align parallel to the applied field.



Consider a polar molecule subjected to an electric field E. The alignment of electric dipole with the electric field is similar to the alignment of magnetic dipole with the applied magnetic field in paramagnetic material.

The expression for polarization can be obtained from the theory of paramagnetism.

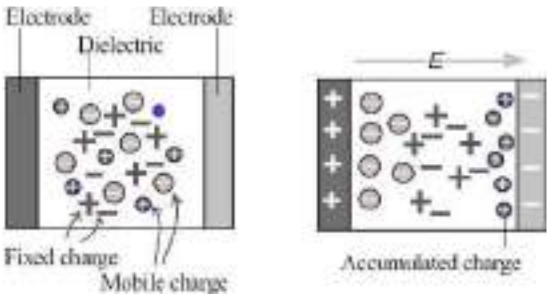
The orientation polarization is given as

$$P_0 = \frac{N\mu_o^2 E}{3KT} = \alpha_o E$$

Where $\alpha_0 = \frac{\mu_o^2}{3KT}$ is called as orientation polarisability

2.8.2.5. Space Charge Polarization:

Space charge polarization occurs due to the accumulation of charges at the electrodes or atinterfaces in a multiphase materials. In the presence of an applied field, the mobile positive ions and negative ions migrate toward the negative electrode and positive electrode respectively to an appreciable distance giving rise to redistribution of charges , but they remain remains in the dielectric material (electrode is blocking). The space charge polarization can be defined as the redistribution of charges due to the applied electric field and the charges accumulate on the surface of the electrodes. It occurs when the rate of charge accumulation is different from rate of charge removal. Space charge polarization is not significant in most of the dielectric materials.



2.9. Internal field in liquids and solids (one dimensional)

In gases state the atoms are separated by large distances and the interaction between the atoms can be neglected. When an external electric field E is applied, the intensity of the electric field experienced by an atom in gases state will be equal to the applied electric field E.

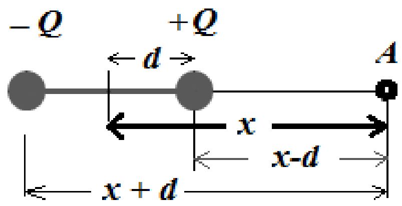
In solids and liquids, the atoms are close to each other leading to strong interaction between them. In solids and liquids the intensity of the electric field at a given point of the material is not equal to the applied electric field but equal to internal field which is the sum of applied electric field and field due to other dipoles present in the material.

$$\text{Internal field} \quad E_i = E + E`$$

The internal field can be calculated by Epstein model in the case of one dimensional atomic array.

2.10. Electric field along the axis of an electric dipole

Consider an electric dipole of length **2d** and charge Q, the field along the axis of the dipole at point A is the sum of the electric field due to +Q and –Q.



The electric field due to +Q at point A is $E^+ = \dfrac{Q}{4\pi\varepsilon_o (x-d)^2}$

The electric field due to +Q at point A is $E^- = \dfrac{Q}{4\pi\varepsilon_o (x+d)^2}$

Electric field of dipole at A is,

$$E_A = E^+ - E^- = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{(x-d)^2} - \frac{1}{(x+d)^2}\right]$$

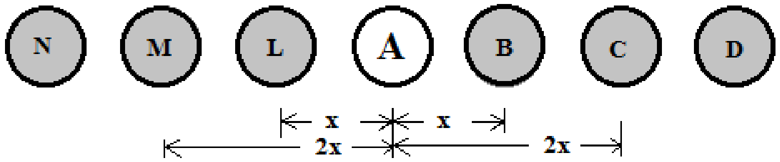
$$= \frac{2Q}{4\pi\varepsilon_o}\left[\frac{2dx}{(x-d)^2(x+d)^2}\right]$$

since $x >> d \quad (x-d)^2 \approx (x+d)^2 \approx x^2$ then

$$= \frac{2Q}{4\pi\varepsilon_o}\left[\frac{2dx}{x^4}\right] = \frac{4dQ}{4\pi\varepsilon_o x^3}$$

since $2dQ = \mu_i$ $\qquad E_A = \frac{2\mu_i}{4\pi\varepsilon_o x^3}$

Consider an array of one dimensional atoms along x- axis. The all the atoms are similar, equally spaced and have induced electric dipole moment μ_i in an applied electric field E. The electric field experienced at the A is the sum of electric fields of other dipoles and applied electric field E.



The electric field atA due to the induced dipole B and L which are at a distance x is

$$E_B = E_L = \frac{2\mu_i}{4\pi\varepsilon_o x^3}$$

The electric field at A due to the induced dipole C and M which are at a distance 2x is

$$E_C = E_M = \frac{2\mu_i}{4\pi\varepsilon_o (2x)^3}$$

Therefore the field due to other dipoles is

$$E' = E_B + E_L + E_C + E_M + E_D + E_N + \ldots.$$

$$E' = \frac{2\mu_i}{4\pi\varepsilon_o x^3} + \frac{2\mu_i}{4\pi\varepsilon_o x^3} + \frac{2\mu_i}{4\pi\varepsilon_o (2x)^3} + \frac{2\mu_i}{4\pi\varepsilon_o (2x)^3} + \frac{2\mu_i}{4\pi\varepsilon_o (3x)^3} + \frac{2\mu_i}{4\pi\varepsilon_o (3x)^3} + \ldots$$

$$E' = \frac{4\mu_i}{4\pi\varepsilon_o x^3} + \frac{4\mu_i}{4\pi\varepsilon_o (2x)^3} + \frac{4\mu_i}{4\pi\varepsilon_o (3x)^3} + \ldots$$

$$E' = \frac{\mu_i}{\pi\varepsilon_o x^3}\left[1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \ldots\right] \quad \text{where } 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \ldots = 1.2$$

$$E' = \frac{1.2\mu_i}{\pi\varepsilon_o x^3}$$

Therefore the internal field E,

$$E_i = E + \frac{1.2\mu_i}{\pi\varepsilon_o x^3}$$

The local field in a three dimensional solid is similar the above equation the number density N of atoms replaces $1/a^3$. Since $N\mu_i = P$ and $1.2/\pi$ is replaced by γ. Then the internal fieldis,

$$E_i = E + \frac{1.2N\mu_i}{\pi\varepsilon_o} = E + \frac{1.2P}{\pi\varepsilon_o} = E + \frac{\gamma P}{\varepsilon_o}$$

γ depends on the internal structure For a cubic symmetry crystal γ value is1/3

$$\boxed{E_i = E + \frac{P}{3\varepsilon_o}}$$

The field given by the above equation is called Lorentz field.

2.11. Clausius - Mosotti equation

Let us consider elemental solid dielectric which exhibits only electronic polarization. If α_e is the electronic polarisability per atom, it is related to the bulk polarization P through the relation

$$P = N\alpha_e E_i$$

$$\alpha_e = \frac{P}{NE_i}$$

Where N is the number of atoms per unit volume and E_i is the local field using the relation above

$$\alpha_e = \frac{P}{N\left[E + \frac{P}{3\varepsilon_o}\right]}$$

By using the relation between the polarization and permittivity we have

$$P = E\varepsilon_0(\varepsilon_r - 1)$$

$$E = \frac{P}{\varepsilon_0(\varepsilon_r - 1)}$$

By Substituting ,

$$\alpha_e = \frac{P}{N\left[\frac{P}{\varepsilon_0(\varepsilon_r - 1)} + \frac{P}{3\varepsilon_o}\right]}$$

$$\frac{N\alpha_e}{\varepsilon_0} = \frac{1}{\left[\frac{1}{(\varepsilon_r - 1)} + \frac{1}{3}\right]} = \frac{1}{\left[\frac{\varepsilon_r + 2}{3(\varepsilon_r - 1)}\right]}$$

The above equation is known as Clausius Mosotti equation which is valid for nonpolar solids

$$\boxed{\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{N\alpha_e}{3\varepsilon_0}}$$

2.12. Dielectric loss:

Dielectric loss is the dissipation of energy through the movement of charges in an alternatingelectromagnetic field as polarisation switches direction.

An efficient dielectric supports a varying charge with minimal dissipation of energy in the formof heat is called dielectric loss. There are two main forms of loss that may dissipate energy within a dielectric.
In conduction loss, a flow of charge through the material causes energy dissipation.

Dielectric loss is especially high around the relaxation or resonance frequencies of the polarisation mechanisms as the polarisation lags behind the applied field, causing an interaction between the field and the dielectric’s polarisation that results in heating. This is illustrated by thediagram below (recall that the dielectric constant drops as each polarisation mechanism becomesunable to keep up with the switching electric field.)

Dielectric loss quantifies a dielectric material's inherent dissipation of electromagnetic energy into, e.g., heat.
It can be represented in terms loss tangent tan δ and is defined:

$$\tan \delta_e = \frac{\epsilon''}{\epsilon'}$$

2.12.1. Dielectric Breakdown : The dielectric breakdown is the sudden change in state of a dielectric material subjected to a very high electric field , under the influence of which , theelectrons are lifted into the conduction band causing a surge of current , and the ability of the material to resist the current flow suffers a breakdown .

Or

When a dielectric material loses its resistivity and permits very large current to flow through it,then the phenomenon is called dielectric breakdown

Or

At high electric fields, a material that is normally an electrical insulator may begin to conductelectricity – i.e. it ceases to act as a dielectric. This phenomenon is known as dielectric breakdown.

2.12.2. Frequency dependence of polarizability:

On application of an electric field, polarization process occurs as a function of time. The polarization P(t) as a function of time. The polarization P(t) as a function of time t is given by

$$P(t) = P[\ 1 - \exp\ (\ - t\ /\ t_r\ )]$$

Where P – max. Polarization attained on prolonged application of staticfield. t_r relaxation time for particular polarization process

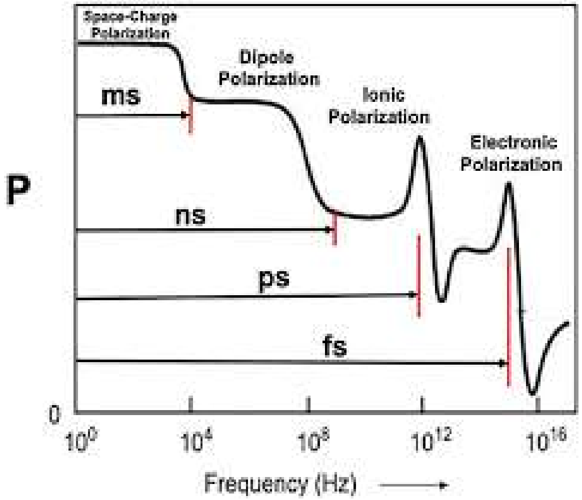
The relaxation time t_r is a measure of the time scale of polarization process. It is the time taken for a polarization process to reach 0.63 of the max. value.

Electronic polarization is extremely rapid. Even when the frequency of the applied voltage is very high in the optical range ($\approx 10^{15}$ Hz), electronic polarization occurs during every cycle of the applied voltage.

Ionic polarization is due to displacement of ions over a small distance due tothe applied field. Since ions are heavier than electron cloud, the time taken for displacement is larger. The frequency with which ions are displaced is of the same order as the lattice vibration frequency (≈1013Hz). Hence, at optical frequencies, there is no ionic polarization. If the frequency of the applied voltage is less than 1013 Hz, the ions respond.

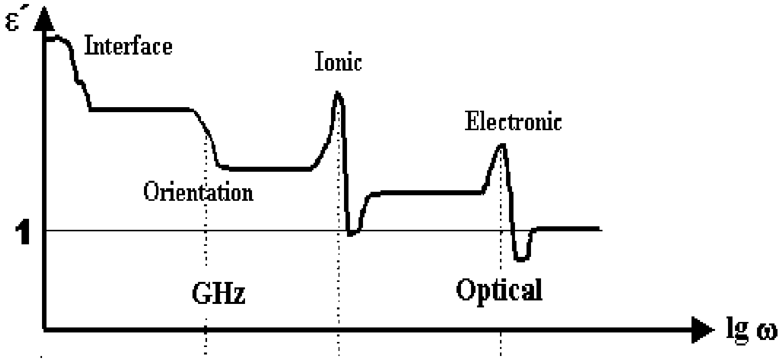
Orientation polarization is even slower than ionic polarization. The relaxation time fororientation polarization in a liquid is less than that in a solid. Orientation polarization occurs, when the frequency of applied voltage is in audio range (1010 Hz).

Space charge polarization is the slowest process, as it involves the diffusion of ionsover several interatomic distances. The relaxation time for this process is related tofrequency of ions under the influence of applied field. Space charge polarization occurs at power frequencies (50-60 Hz).



2.12.3. Frequency Dependence of dielectric constant

When a dielectric material is subjected to an alternating field, the polarization component required to follow the field in order to contribute to the total polarization of the dielectrics. The relative permittivity which is a measure of the polarization also depends on the frequency. The dependence of $\square_r$ on frequency of the electric field is shown in the figure.



At very low frequency, the dipoles will get sufficient time to orient themselves completely with the field and all types of polarization exist. Since the dielectric is characterized by polarisability $\alpha\ (\alpha = \alpha_e + \alpha_i + \alpha_o$ at low frequency i.e at radiofrequency region the dielectricconstant will be due to all polarisability.

The orientation polarization, which is effective at low frequencies, is damped out for higher frequencies. In the microwave region the dipoles fail to follow the field and the polarisability reduces to ($\alpha = \alpha_e + \alpha_i$), as a result ε_r decreases to some amount.

In the IR region the ionic polarization fails to follow the field so the contribution of ionic polarization dies away. In this region only electronic polarization contributes to the total polarization. Therefore ($\alpha = \alpha_e$) the ε_r still decreases and only electronic polarization exist.

We know that

$$P = N\alpha_e E = \varepsilon_o(\varepsilon_r - 1)E$$

Then, the relative permittivity is,

$$\varepsilon_r = 1 + \frac{N\alpha_e}{\varepsilon_o}$$

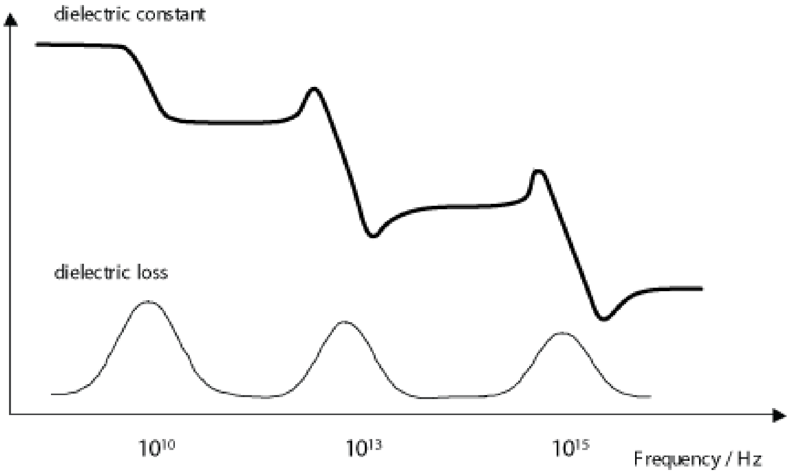
In the ultraviolet region even the electron cloud could not follow the field and electronic polarizability becomes almost zero and the permittivity becomes one.

$$[\varepsilon_r]_{X-ray} = 1$$

For example at low frequency the dielecric constant of water at room temperature is about 80, but it fall to about 1.8 in the optical region.

2.12.4. Frequency Dependence of dielectric loss:

Dielectric loss tends to be higher in materials with higher dielectric constants. This is the downside of using these materials in practical applications. Dielectric loss is utilised to heat foodin a microwave oven: the frequency of the microwaves used is close to the relaxation frequency of the orientational polarisation mechanism in water, meaning that any water present absorbs a lot of energy that is then dissipated as heat. The exact frequency used is slightly away from the frequency at which maximum dielectric loss occurs in water to ensure that the microwaves are not all absorbed by the first layer of water they encounter, therefore allowing more even heating of the food.



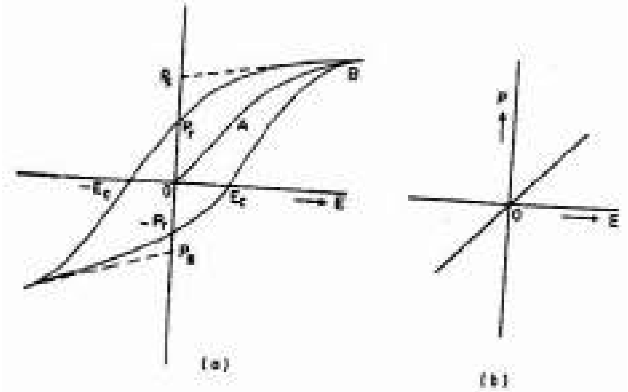
2.13. Ferroelectrics

Below certain temperature it is found that some materials spontaneously acquire an electric dipolemoment. These materials are called as ferroelectric materials or ferroelectrics.The temperature at which ferroelectric property of the material disappears is called as ferroelectric Curie temperature.

Ferroelectric materials are anisotropic crystals which exhibit a hysteresis curve P versusE which can be explained by domain hypothesis.

- **Ferro electricity:** Ferro electric materials are an important group not only because ofintrinsic Ferro electric property, but because many possess useful piezo electric, birefringent and electro optical properties.

The intrinsic Ferro electric property is the possibility of reversal or change of orientation of the polarization direction by an electric field. This leads to hysteresis in the polarization P, electric field E relation , similar to magnetic hysteresis. Above a critical temperature, the Curie point T_c, the spontaneous polarization is destroyed by thermal disorder.The permittivity shows a characteristic peak at $T_{c.}$



- **Piezo – Electric Materials and Their Applications:** Single crystal of quartz is used for filter, resonator and delay line applications. Natural quartz is now being replaced by syntheticmaterial.

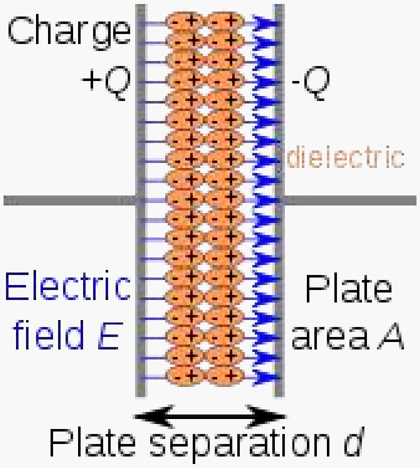
Rochelle salt is used as transducer in gramophone pickups, ear phones, hearing aids, microphones etc. the commercial ceramic materials are based on barium titanate, lead zirconate and lead titanate. They are used for high voltage generation (gas lighters), accelerometers, transducers etc. Piezo electric semiconductors such as GaS, ZnO & CdS are used as amplifiers of ultrasonic waves.

2.14. Applications of Dielectric Materials:

Almost any type of electrical equipment employs dielectric materials in some form or another. Wires and cables that carry electrical current, for example, are always coated or wrapped with some type of insulating (dielectric) material. Sophisticated electronic equipment such as rectifiers, semiconductors, transducers, and amplifiers contain or are fabricated from dielectric materials. The insulating material sandwiched between two conducting plates in a capacitor is also made of some dielectric substance.

Liquid dielectrics are also employed as electrical insulators. For example, transformer oil is a natural or synthetic substance (mineral oil, silicone oil, or organic esters, for example) that has the ability to insulate the coils of a transformer both electrically and thermally.

2.14.1. Capacitors



Commercially manufactured capacitors typically use a solid dielectric material with high permittivity as the intervening medium between the stored positive and negative charges. This material is often referred to in technical contexts as the capacitor dielectric.

The most obvious advantage to using such a dielectric material is that it prevents the conducting plates, on which the charges are stored, from coming into direct electrical contact. More significantly, however, a high permittivity allows a greater stored charge at a given voltage. This

$$\sigma_\varepsilon = \varepsilon \frac{V}{d}$$

can be seen by treating the case of a linear dielectric with permittivity ε and thickness d between two conducting plates with uniform charge density σ_ε. In this case the charge density is given by

and the capacitance per unit area by

$$c = \frac{\sigma_\varepsilon}{V} = \frac{\varepsilon}{d}$$

From this, it can easily be seen that a larger ε leads to greater charge stored and thus greater capacitance.

Dielectric materials used for capacitors are also chosen such that they are resistant to ionization. This allows the capacitor to operate at higher voltages before the insulating dielectric ionizes and begins to allow undesirable current.

2.14.2. Dielectric resonator

A dielectric resonator oscillator (DRO) is an electronic component that exhibits resonance of the polarization response for a narrow range of frequencies, generally in the microwave band. It consists of a "puck" of ceramic that has a large dielectric constant and a low dissipation factor. Such resonators are often used to provide a frequency reference in an oscillator circuit. An unshielded dielectric resonator can be used as a Dielectric Resonator Antenna (DRA).

2.14.3. Insulators-

Required Qualities of Good Insulating Materials: The required qualities can be classified as under electrical, mechanical, thermal and chemical applications.

2.14.4. Electrical:

1. electrically the insulating material should have high electrical resistivity and high dielectric strength to withstand high voltage.
2 .The dielectric losses must be minimum.
3. Liquid and gaseous insulators are used as coolants. For example transformer oil, Hydrogen and helium are

2.14.5. Mechanical:

1. insulating materials should have certain mechanical properties depending on the use to which they are put.
2. When used for electric machine insulation, the insulator should have sufficient mechanical strength to withstand vibration.

2.14.6. Thermal:

Good heat conducting property is also desirable in such cases. The insulators should have small thermal expansion and it should be non-ignitable.

2.14.7. Chemical:

1. chemically, the insulators should be resistant to oils, liquids, gas fumes, acids and alkali’s.

2. The insulators should be water proof since water lowers the insulation resistance and the dielectric strength.

2.14.8. Other Applications:

Solid dielectrics are perhaps the most commonly used dielectrics in electrical engineering, as very good insulators. Some examples include porcelain, glass, and most plastics.
Air, nitrogen and sulfur hexafluoride are the three most commonly used gaseous dielectrics.

Industrial coatings such as parylene provide a dielectric barrier between the substrate and its environment.

Mineral oil is used extensively inside electrical transformers as a fluid dielectric and to assist in cooling. Dielectric fluids with higher dielectric constants, such as electrical grade castor oil, are often used in high voltage capacitors to help prevent corona discharge and increase capacitance.

Because dielectrics resist the flow of electricity, the surface of a dielectric may retain stranded excess electrical charges. This may occur accidentally when the dielectric is rubbed (the triboelectric effect). This can be useful, as in a Van de Graaff generator or electrophorus, or it can be potentially destructive as in the case of electrostatic discharge.

Piezoelectric materials are another class of very useful dielectrics which are used for transducers and sensors.
Ferroelectric materials often have very high dielectric constants, making them quite useful forcapacitors.

2.15. Magnetic Materials

Classification of dia, para and ferromagnetic materials. Curie Temperature Hysterisis in ferromagnetic materials. Soft and Hard magnetic materials. Applications.

Introduction

The materials that can be magnetised are called as magnetic materials

- Magnetic dipoles and magnetic dipole moment

Any two opposite poles separated by distance constitute an magnetic dipole. A magnet is a dipole which has north pole and south pole and the length of the magnet is the distance of separation.

Magnetic dipole moment is the product of magnetic pole strength (m) and length of the magnet(l)

- Magnetic field intensity (H)

$$\mu_m = m.l$$

The force experienced by a unit north pole (of strength 1 Wb) placed at a point in a magnetic field is a measure of the` field intensity' or `field strength'

- Magnetisation or Intensity of magnetisation (M)

Magnetization may be defined as the process of converting a non magnetic bar into a magnetic bar

- Magnetic Induction Or Flux Density(B)

Magnetic induction or magnetic flux density in an any material is the number of lines of magnetic force passing through unit area perpendicular . Wb/m^2

$$B = \frac{\phi}{A} = \mu_0(M + H)$$

- Magnetic Susceptibility (χ)

The ratio of the magnetization to the field strength

$$\chi = \frac{M}{H}$$

- Permeability (μ)

The ration of the amount of magnetic density B to the applied magnetic field . It is used to measure magnetic lines of forces passing through the material

$$\mu = \frac{B}{H}$$

- Origin of Magnetic moment – Bohr Magneton

When ever a charged particle has an angular momentum , it contributes to permanent dipole moment. Consider an hydrogen atom, electron revolving around the nucleus is equal to a current loop. Orbital angular momentum arises due this current loop. The electron spin angular momentum and nuclear spin angular momentum arises due to spin of the electron and nucleus respectively.

There are three angular momentum of an atom

1. Orbital angular momentum of the electron
2. Electron spin angular momentum
3. Nuclear spin angular momentum

Total angular magnetic momentum

Bohr Magneton:

The orbital angular momentum of an electron in an atom can be expressed in terms of atomic unit of magnetic moment called Bohr Magneton.

$$\mu_B = \frac{eh}{4\pi m}$$

2.16. Classification of the magnetic materials

The magnetic materials are broadly classified in to two types. They are 1. Those atoms or molecules do not have permanent dipole moments and 2. Those atoms or molecules have permanent dipole moments even in the absence of external magnetic field.

Based on the magnetic moments the materials are classified as

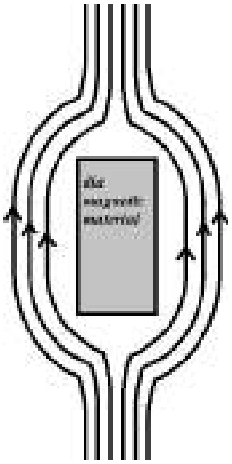
Diamagnetic materials
Paramagnetic materials
Ferromagnetic materials
Anti ferromagnetic materials
Ferri magnetic material

2.16.1. Diamagnetic materials

Dia magnetic materials has completely filled sub shell electronic structure resultant magnetic moment is zero. There are no permanent dipoles and hence the magnetic effect are small. When a diamagnetic material is placed in a magnetic field, there will be a small induced magnetic moment which always oppose the applied field(accordance with Lenz`s Law). Due to this effect the magnetic lines of forces expelled from the materials.

Mostly the covalent and ionic crystals exhibits the diamagnetic properties. The magnetic susceptibility is small and negative and is independent of temperature.

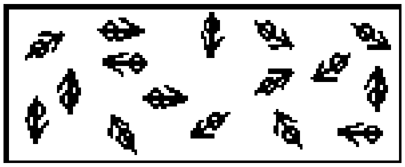
The examples of diamagnetic materials are 1. Covalent materials such as Si,Ge, diamond, ii) some metals such as copper, silver, gold.

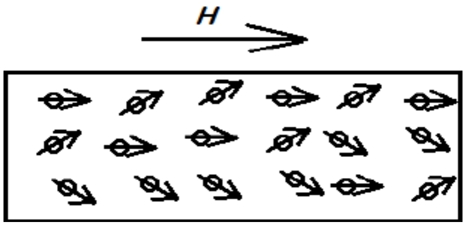


2.16.2. Para magnetic materials.

Atoms or molecules of paramagnetic materials have permanent magnetic moment oriented in random direction. The magnetic interaction between the dipoles try to align themselves but the thermal agitation disturb the alignment. In paramagnetic materials vector sum of magnetic moments is zero in the absence of field. When an external magnetic field is applied the partial alignment of permanent atomic magnetic moments occur

When a magnetic field is applied , the individual magnetic moment takes the alignment along the applied field as shown in figure . The magnetization of a paramagnetic material increases with the increase in the applied field. Increase in temperature it reduces the magnetization and destroys the alignment of dipoles with applied field.





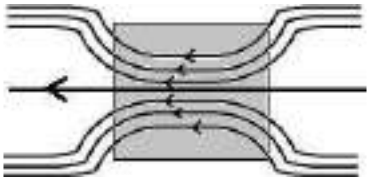
Consider a paramagnetic material placed in non-uniform magnetic field. The paramagnetic materials experience a net magnetic force towards the greater field. The magnetic susceptibility is small and positive and is dependent on temperature. The susceptibility of the magnetic field is given by

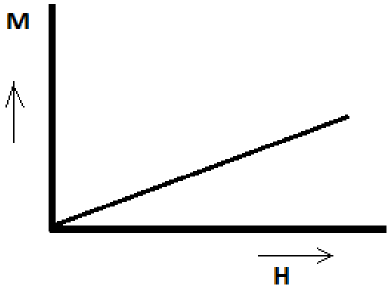
$$\chi = \frac{C}{T}$$

Where C is the curie temperature and T is the temperature in Kelvin scale.

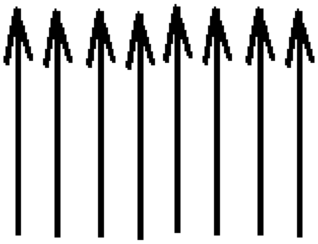
The magnetization in ferromagnetic material is linear and gets saturated when a large magnetic field is applied at low temperature.

The examples of paramagnetic materials are Mg, gaseous and liquid oxygen , ferromagnetic material (Fe), and anti-ferromagnetic materials at high temperature and ferromagnetic material (Fe_3O_4) at high temperature.





2.16.3. Ferromagnetic materials



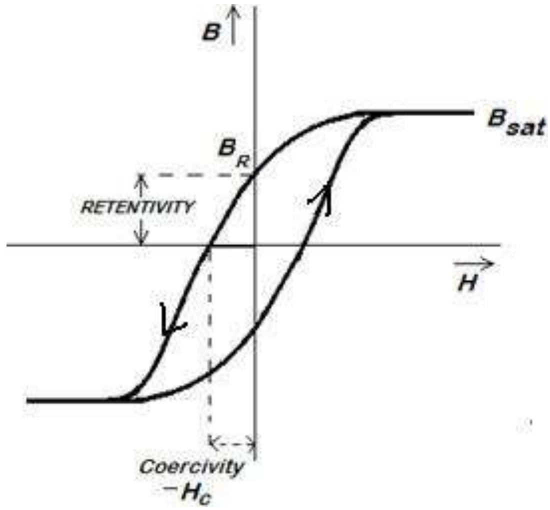
Atoms or molecules of ferromagnetic materials have permanent magnetic moment. In ferromagnetic materials all the dipoles are aligned parallel as shown in the figure if a small value of magnetic field is applied, a large value of magnetization is produced. As the ferromagnetic material have permanent magnetic dipole moment and the susceptibility is positive. The magnetization in ferromagnetic material is non linear and gets saturated when a large magnetic field is applied.

A ferromagnetic materials exhibits two different properties. It behaves as a ferromagnetic material below a certain temperature known as ferromagnetic **curie temperature**. Above the temperature it behaves as a paramagnetic material. In the ferromagnetic region, it exhibits well known curve known as hysteresis curve as shown in the figure.

The susceptibility of a ferromagnetic material above the ferromagnetic curie temperature is given by

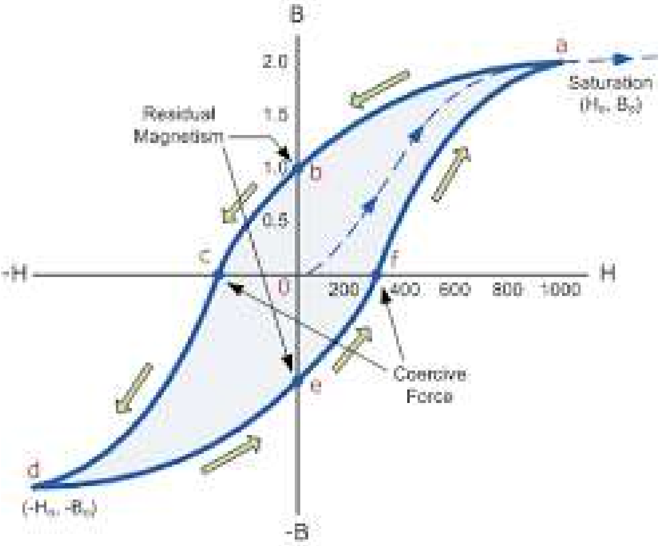
$$\chi = \frac{C}{T - \theta_f}$$

Where C is the Curie constant and θ_f is the ferromagnetic Curie temperature. The transition and rare earth metals such as Fe ,Co, Ni ,Gd are the examples of ferromagnetic materials.



2.16.4. Hysteresis in ferromagnetic materials (B-H curve)

Below the ferromagnetic Curie temperature ($T < \theta_f$) Ferromagnetic material exhibits a well known curve called hysteresis curve. The variation of B(magnetic induction) with H(applied field) can be represented by a closed curve called hysteresis loop or curve. This refers lagging of magnetization behind the magnetising field. If a magnetic field is increased gradually, the flux density increases and it becomes maximum. The maximum value of flux density is called saturated magnetization.If the field is reversed, the ferromagnetic materials is found to have magnetization in the absence opf external field. This is called as retentivity or remanent magnetization (B_R) and this property is called as spontaneous magnetization. If the field is further reduced the flux density reduces to zero. The field requird in the opposite direction to bring magnetization to zero is called as coercive field orcoercivity ($-H_c$). If the field is increased in oposite direction it attains saturation magnetization. Ifan alternating field is applied a closed loop as shown in the figure is obtained.



According to Weiss, a virgin specimen of ferromagnetic material consist of number of regions or domains($\approx 10^{-6}$ m or above) which are spontaneously magnetized. When magnetic field is not applied the direction of spontaneous magnetization varies from domain to domain. The resultant magnetization may hence be zero or nearly zero.

These domains are separated from other by a wall known as domain wall or Bloch wall The domain concept is used to explain the hysteresis property. When an external field is applied two possible ways of alignment domain growth are possible one by domain wall motion and other byrotation of domain wall and domain growth is also reversible. Hysteresis curve is explained by domain concept.

- **Antiferromagnetic matériels :** These are the ferromagnetic materials in which equal no of opposite spins with same magnitude such that the orientation of neighbouring spins is inantiparallel manner are present.

Susceptibility is small and positive and it is inversely proportional to the temperature.

$$\chi = C/(T+\theta)$$

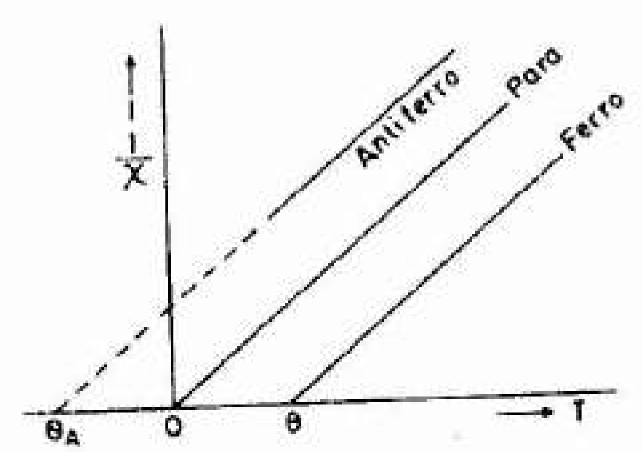
the temperature at which anti ferromagnetic material converts into paramagnetic material is known as Neel's temperature.

Examples: FeO, Cr_2O_3.

- **Ferrimagnetic materials:** These are the ferromagnetic materials in which equal no of opposite spins with different magnitudes such that the orientation of neighbouring spins is in antiparallel manner are present.

Susceptibility positive and large, it is inversely proportional to temperature

$\chi = C/(T \pm \theta)$ T> T_N (Neel'stemperature) Examples : $ZnFe_2O_4$, Cu

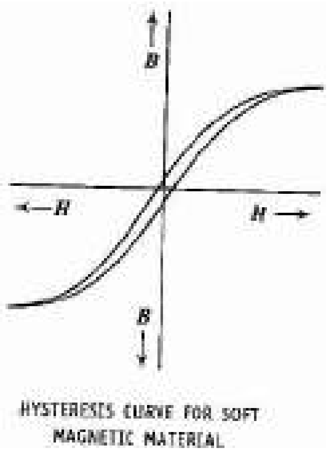


2.16.5. Soft and Hard magnetic materials

- Soft magnetic materials

The magnetic materials that are easy to magnetize and demagnetize are called as soft magnetic materials.

Properties:



1. Low remanent magnetization
2. Low coercivity
3. Low hysteresis energy loss
4. Low eddy current loss
5. High permeability
6. High susceptibility

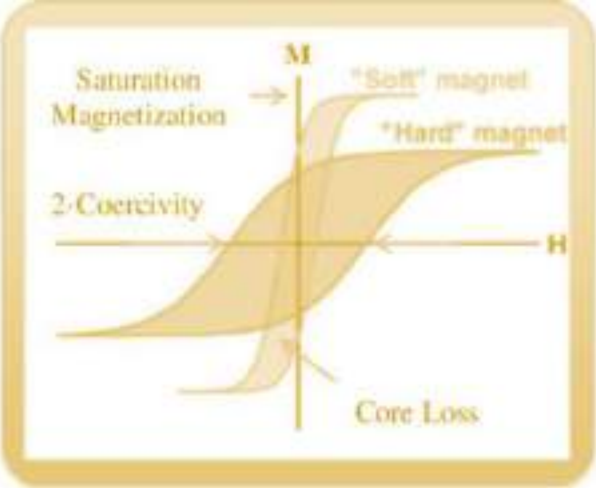
Examples of soft magnetic materials are

i) Permalloys (alloys of Fe and Ni)
ii) Si – Fe alloy
iii) Amorphous ferrous alloys (alloys of Fe, Si, and B)
iv) Pure Iron (BCC structure)

Applications of soft magnetic materials: Mainly used in electro- magnetic machinery and transformer cores. They are also used in switching circuits, microwave isolators and matrix storage of computers.

- *Hard magnetic materials*

The magnetic materials that are difficult to magnetize and demagnetize are called as hardmagnetic materials.



Properties:

1. High remanent magnetization
2. High coercivity
3. High saturation flux density
4. Low initial permeability
5. High hysteresis energy loss
6. High permeability
7. The eddy current loss is low for ceramic type and large for metallic type.

Examples of hard magnetic materials are, i) Iron- nickel- aluminum alloys with certain amount of cobalt called Alnico alloy. ii) Copper nickel iron alloys. iii) Platinum cobalt alloy.

Applications of hard magnetic materials: For production of permanent magnets, used in magnetic detectors, microphones, flux meters, voltage regulators, damping devices and magnetic separators.

Hard magnetic materials	Soft magnetic materials
Difficult to magnetize and demagnetize	Easy to magnetize and demagnetize
large hysteresis loop area	small hysteresis loop area
Have large hysteresis loss	Have very low hysteresis loss
The domain wall movement is difficult and it is irreversible in nature	Domain wall movement is relatively easier. Even for small change in the magnetizing field ,magnetization changes by large amount
The coercivity and retentivity are large	The coercivity and retentivity are small
Magnetostatic energy is large	Magnetostatic energy is small.
Small values of permeability and susceptibility	Large values of permeability and susceptibility
Used to make permanent magnets Examples- Iron-nickel-aluminium alloys (alnicol) Copper nickel iron (cunife)	Used to make electromagnet Examples- Fe-Si , Ferrous nickel alloys ,Ferrites,Garnets

Possible Numerical Questions:

1. Find the capacitance of a layer of Al_2O_3that is 0.5 m thick and 2000 mm^2 of square area ($\varepsilon_r = 8.854 \times 10^{-12}$ Fm^{-1})
2. A layer of porcelain is 80 mm long, 20 mm wide and 0.7 µm thick. Calculate its capacitance with $\varepsilon_r = 6$
3. If an ionic crystal is subjected to an electric field of 1000 Vm^{-1} and the resulting polarization 4.3×10^{-8} cm^2. Calculate the relative permittivity of NaCl. Solution:
4. Calculate the electronic Polarization of argon atom. Given at $_r = 1.0024$ at NTP and $N = 2.7 \times 10^{25}$ atom m^{-3} .
5. A solid contains 5×10^{28} atoms/m^3 each with a polarisability of 2×10^{-40} F m^2. Assuming that the internal field is given by Lorentz formula. Calculate the ratio of internal field to the external field. $\varepsilon_0 = 8.854 \times 10^{-12}$ Fm^{-1}.
6. Calculate the polarization produced in dielectric medium of dielectric constant 6 when it is subjected to an electric field of 100 Vm–1. (ε_0 = 8.854 × 10–12 Fm^{-1})
7. Calculate the electronic polarizability of neon. The radius of neon atom is 0.158 nm
8. The dielectric constant of a helium gas at NTP is 1.0000684. Calculate the electron polarizability of helium atoms if the gas contains 2.7×10^{26} atoms/m^3 and hence calculate the radius of helium atom (= 8.854×10^{-12} Fm^{-1})

Model sample questions:

1. What is dielectric material? Mention the uses of dielectric materials.
2. Define the following terms (i) Dielectric Polarization, (ii) Polarisability, (iii) DielectricConstant, (iv) Spontaneous polarization, (v) Electric susceptibility.
3. Establish the relation concerning **D**, **E**, and **P**.
4. What do you mean by Macroscopic and Microscopic field? What is Lorentz field?
5. Briefly discuss the polarization phenomenon in dielectric materials.
6. Explain briefly the term electronic, ionic and orientation polarization in dielectric system under static electric field. Derive Langevin-Debye expression
7. What is relative dielectric constant? Derive the expression for the dielectric susceptibility.Briefly

describe with the help of atomic model the dependence of relative dielectric constant (ε_r) on the electronic polarizability (α_e).

8. Is the relative dielectric constant (ε_r) depends on the atomic size of dielectric materials? Explain.
9. What do you mean by orientational polarization? Discuss the temperature dependence of orientational polarization.
10. Estimate the internal electric field strength with in the atoms of dielectric material, placedunder external static electric field.
11. What is Clausius-Mosotti relation of dielectric constant? Develop the relation for elemental dielectric systems.
12. Arranged in descending order the relative dielectric constant of following materials :
    (a) LiCl (b) Ge (c) Diamond (d) Si
13. What should be the effect on relative dielectric constant of ionic solid materials, whenplaced under optical radiation.
14. Short note: (a) dielectric strength, (b) dielectric loss, (c) Ferroelectricity, (d) Piezoelectricity.
15. Give name of 3 each – Ferro and Piezo- electric materials. Also point out their mostimportant uses in industry.
16. Discuss the frequency effect on dielectric value of material.
17. Discuss the frequency dependence of various contributions to dielectric polarizability.
18. Explain briefly the behavior of electronic polarizability (α_e) of dielectric material under a.c field application.
19. Compare the values of natural angular frequencies in (i) electronic polarization and (ii)ionic polarization of dielectric material under a.c field application.
20. What do you mean by complex dielectric constant of material?
21. Write down the expressions of following parameters under static and a.c. field application of dielectric materials :
    (a) Internal electric field (Ei)
    (b) Electric dipole (P)
    (c) Relative dielectric constant
22. What is dielectric loss? Define loss-tangent of dielectric.
23. Write down the equivalent circuit of a loss-dielectric capacitor.
24. What is dielectric strength? Compare the same in (a) mica, (b) fused silica (c) porcelain.Arrange then in ascending order.

25. What should be the nature of relative dielectric constant (ε_r) under A.C. field? How it isrelated with the dielectric loss in the system?

CHAPTER 3:

QUANTUM PHYSICS

Chapter - 3

In the late 19th and early 20th centuries Quantum theory emerged from a series of revolutionary discoveries which pointed out the limitations of classical physics and developed the foundation for understanding the behavior of matter and energy at atomic and subatomic scales. This chapter deals with the key experiments and theoretical advances, that shaped the progress of quantum mechanics.

3.1. Inadequacy of Classical Physics and the Requirement of Quantum Theory

Newtonian mechanics, Maxwell's electromagnetic theory, and thermodynamics are the key components of Classical physics which can easily describe the macroscopic physical world. However, with the progress and development of the modern science, several phenomena at the atomic and subatomic realms found to be inexplicable in terms of classical physics. To overcome these difficulties, the quantum theory was developed.

3.1.1 Blackbody Radiation and the Ultraviolet Catastrophe

According to the classical Rayleigh-Jeans law the energy density of blackbody radiation in the range of wavelength λ and λ+dλ is given by –

$$E_\lambda d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda$$

Which suggest that as $\lambda \to 0, E_\lambda \to \infty$ that is in huge contrast with the experimentally obtained result which suggests that $\lambda \to 0, E_\lambda \to 0$. This inconsistency of experimental and theoretical result is known as ultraviolet catastrophe.

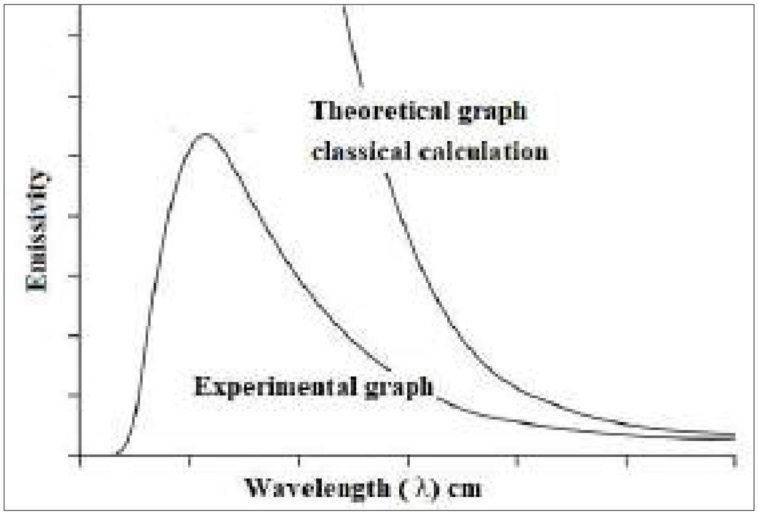


Fig: The experimental curve and the theoretical graph obtained by Rayleigh – Jeans law.

3.1.2 Photoelectric Effect

The ejection of electrons from a metallic surface due to incidence of electromagnetic radiation of high frequency is termed as photoelectric effect.

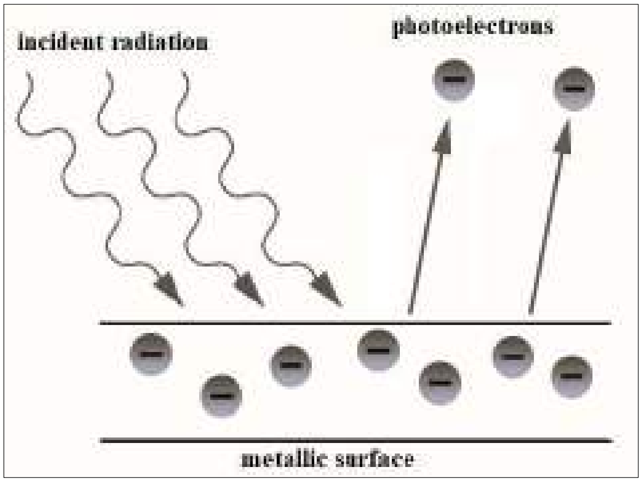


Fig: Schematic diagram of photoelectric effect

Classical electromagnetic theory of light fails to explain why the incident of photoelectron emission is instantaneous in nature as the classical theory predicts requirement of minimum time for transfer of energy from incident photons to the atoms of the metal. Experimentally it has been observed that the photocurrent is proportional to the intensity of the radiation and kinetic energy of the emitted electrons is dependent upon the frequency of incident radiation. However, the electromagnetic theory could not explain it clearly. Moreover, the classical theory fails to explain the existence of a threshold frequency for a material to show photoelectric effect.

3.1.3 Stability of Atom

According to Rutherford’s model of atomic structure, negatively charged electrons are revolving around the positively charged heavy nucleus in circular orbit. The classical electromagnetic theory states that an accelerating charge particle will spontaneously emit energy. Therefore, with course of time the electrons will lose energy and after a certain time will collapse into the nucleus. Thus, the Rutherford’s model fails to ensure the stability of nucleus.

3.1.4 Absorption and Emission Spectra of Atoms

Classical theory predicts that the atoms emit or radiates energy continuously having all types of wavelengths. However, discrete spectral lines have been observed in the emission and absorption spectra of different atoms which indicates that only certain definite electronic transitions are possible and which further strengthen the fact that the classical physics is not sufficient for calculation in atomic level.

3.1.5 Compton Effect

It was predicted by classical electromagnetic theory that the wavelength of a scattered X-ray by electrons inside a material remains the same as the incident X-rays. Then Arthur Compton's experiments revealed that the wavelength of the scattered X-ray becomes longer compared to the incident X-rays. It has also been observed that the increase of wavelength depends on the scattering angle which indicates particle like behavior of X-ray. Classical theory failed to describe it.

3.1.6 Diffraction and Interference of Electrons

The famous double slit experiment of electron shows the presence of interference pattern indicating the wave nature of electron, while according to the classical theory electrons should behave like particle and there will be two distinct band in the screen. This type of wave-particle dual nature of electron was successfully explained by quantum theory.

Many similar types of examples are available in the history of physics where classical physics was not sufficient to explain the phenomena and this inadequacy was successfully addressed by the Quantum theory.

3.2. Black Body Radiation:

Blackbody is an idealized physical object which can absorb all the radiation incident upon it. The black body radiation is function of its temperature and is characterized by a continuous spectrum that varies with temperature. Some definitions which will be helpful for the study of black body radiation are –

3.2.1 Emissive power (e_λ):

The emissive power (e_λ) of an object at a particular temperature and wavelength is defined as the total amount of energy emitted normally per unit surface area of the body per unit time within unit range of wavelength.

3.2.2 Absorptive power (a_λ):

Absorptive power is a measure of the ability of a material to absorb radiation incident upon it.

Absorptive power is defined as the ratio of the amount of energy absorbed by a body to the total amount of incident radiation energy on the surface –

$$a_\lambda = \frac{Q_{absorbed}}{Q_{incident}}$$

It is a dimensionless quantity and its value ranges from 0 to 1.

3.2.3 Kirchhoff's law of thermal radiation:

Kirchhoff's Law of Thermal Radiation states that for a given wavelength and temperature, the ratio of the emissive power of a body to its absorptive power is constant. Mathematically, it is expressed as:

$$\frac{e_\lambda}{a_\lambda} = constant$$

For a perfect black body, the absorptive power is 1. Which suggests that for black body radiation $e_\lambda = constant$

3.2.4 What is a black body?

A material which can absorb all the incident electromagnetic radiation irrespective of frequency or angle of incidence and does not reflect or transmit any radiation is known as perfect black body. It is well known that a good absorber of radiation is also a good emitter which therefore suggests that a perfect blackbody can emit radiation of all wavelength when it is heated.

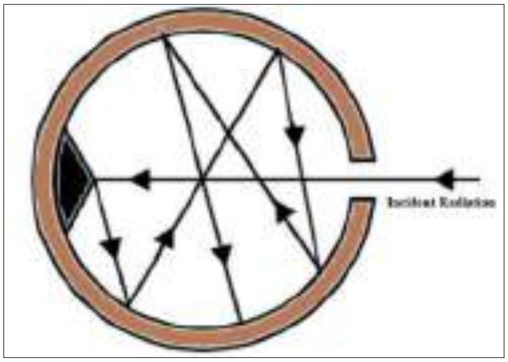


Fig: Schematic Diagram of a Black Body

To successfully construct a blackbody one must be very careful to design the cavity and choice the materials to ensure high emissivity and minimum loss of energy. The schematic diagram of a perfect black body has been shown in the above figure where an enclosure has been constructed with a small opening in one side. The interior walls of the enclosure have been coated with a material which will help minimizing reflections and maximize the absorption of incident radiation. A radiation which enters the spherical enclosure suffers multiple number of reflections in the inner surface of the walls and after a multiple number of reflections it completely get absorbed.

3.2.5. Spectrum of black body radiation

The energy distribution spectrum of a blackbody is the graph between emissive power and wavelength at a particular temperature.

Key Features of the Blackbody Radiation Spectrum:

i**.** The blackbody radiation spectrum is continuous and for a particular temperature, the emissive power increases with the increase of wavelength and attains a maximum value for a particular value of wavelength λ_m. On further increase of wavelength, the emissive power decreases rapidly.

ii. The intensity of radiation emitted by a blackbody peak at a specific wavelength (λ_{max}), which depends on the temperature of the blackbody. According to Wien's displacement law, the peak wavelength is inversely proportional to the temperature.

iii. It has been observed experimentally that the wavelength for which the emissivity is maximum at a particular temperature shift towards lower wavelength region upon increase of the temperature.

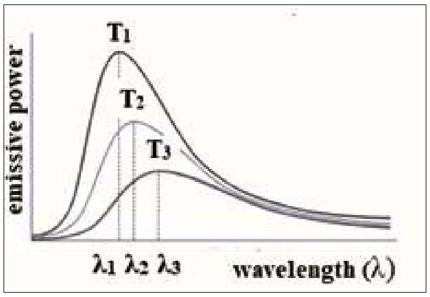


Fig: Variation of emissive power with the wavelength at different temperature ($T_1>T_2>T_3$)

3.2.6. Wien's displacement law: In 1893, Wilhelm Wien proposed this law to understand the relationship between the peak wavelength (λ_{max}) where the energy density of the blackbody radiation is maximum and the corresponding temperature (T) of the blackbody. With a simple mathematical expression this law successfully shows how the location of the maximum energy density of the blackbody radiation shifts as the temperature is changed. If λ_{max} is the wavelength at maximum energy distribution of a black body radiation at a particular temperature (T), then –

$$\lambda_{max}T = constant$$

This law suggests that the maximum energy density of the radiation emitted from a black body get shifted towards the lower wavelength region for the rise of the temperature (T) of the black body.

3.2.7. Theoretical explanation of Black body spectrum:

Initially scientists attempted to to explain blackbody radiation using classical electromagnetic theory and statistical mechanics. Here initially two well-known classical models have been discussed which appears to describe the black body radiation spectrum partially. Next we have discussed the famous Planck’s hypothesis and radiation law which clearly explain the experimentally obtained blackbody spectrum throughout whole wavelength region.

3.2.7.1 Wien’s radiation formula:

In order to explain the observed blackbody spectrum, Wien first showed that the energy density of radiation of wavelength λ and λ+dλ from a black body of temperature T is –

$$E_\lambda d\lambda = \frac{A}{\lambda^5} e^{-\frac{B}{\lambda T}} d\lambda$$

Here, A and B are constant. This law explains the experimentally observed black body spectrum for the low wavelength region. But for high wavelength region, the result is not satisfactory.

3.2.7.2 Rayleigh-Jeans Law:

Lord Rayleigh and James Jeans used classical equipartition of energy theorem along with the other classical calculations to study the nature of black body radiation. They found, the energy density of radiation in the wavelength range λ and λ+ dλ from a black body at a temperature T is given by:

$$E_\lambda d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda$$

Few assumptions were considered by Lord Rayleigh and James Jeans in order to explain the experimentally obtained spectrum of blackbody radiation. The assumptions are –

(i) The black body cavity is full of radiations of all frequencies starting from 0 to ∞. The radiations are composed of electromagnetic wave which gets reflected by the walls of the container forming stationary waves.

(ii) The number of modes per unit volume of the cavity within the frequency range ν and ν+dν is given by $n_\nu d\nu = 8\pi\nu^2 d\nu/c^3$, where c is the velocity of light in free space.

(iii) The average energy per degree of freedom is $1/2kT$, where k is the Boltzmann constant (1.38 × 10^{-23} J/K) according to the Maxwell's law of equipartition of energy.

The electromagnetic standing wave inside the cavity of the black body can be considered as linear harmonic oscillator. Considering the kinetic and potential energy, the energy of each vibration will be –

$$\bar{\epsilon} = 2 \times \frac{1}{2} kT = kT$$

Thus, the energy density (ie. the amount of radiant energy per unit volume) within the frequency range ν and ν+dν is given by –

$$E_\nu d\nu = \bar{\epsilon}(n_\nu d\nu) = kT\frac{8\pi\nu^2}{c^3}kTd\nu$$

Again using,$\nu = \frac{c}{\lambda}$; that is $d\nu = \left|-\frac{c}{\lambda^2}d\lambda\right|$ so the above equation can be written in terms of wavelength as –

$$E_\lambda d\lambda = \frac{8\pi kT}{\lambda^4}d\lambda$$

The Rayleigh-Jeans law explains the experimental result at high wavelength region but fails completely for low wavelength region.

Ultraviolet catastrophe:

The ultraviolet catastrophe describes a problem that arose in classical physics at the end of the 19th century. This prediction was in stark contrast with experimental observations and is considered a major problem that led to the development of quantum mechanics.

According to the Rayleigh-Jeans law,

$$E_\lambda d\lambda = \frac{8\pi kT}{\lambda^4}d\lambda$$

According to this relation $E_\lambda \rightarrow \propto$ when $\lambda \rightarrow 0$. But according to the experimental results $E_\lambda \rightarrow 0$ when $\lambda \rightarrow 0$. This difference between theoretical results obtained from the Rayleigh-Jeans law and the experimentally obtained graph is termed as ultraviolet catastrophe.

Planck's resolution of the ultraviolet catastrophe marked the birth of quantum theory. It showed that classical physics could not explain all physical phenomena and that a new framework was needed to understand the behavior of energy and matter on very small scales. This ultimately led to the development of quantum mechanics, which has become a fundamental theory in modern physics.

3.2.7.3 Planck’s radiation law:

Planck’s radiation law was derived by adopting the following steps –

1. An enclosure of finite volume has been considered where every atoms can emit blackbody radiation and the atoms in the wall of black body radiator behaves like simple harmonic oscillator.

2. Density of States: The number of possible standing wave modes per unit volume within the frequency range ν and ν+dν is given by

$$n_\nu d\nu = 8\pi\nu^2 d\nu / c^3$$

3. According to Planck's hypothesis, the energy of each mode is quantized and given by $E = nh\nu$, n is a non-negative integer and h is Planck's constant.

4. Classical Boltzmann distribution law has been followed to calculate average energy.

The average energy of a mode is –

$$\langle E \rangle = \sum_{n=0}^{\infty} \frac{nh\nu e^{-\frac{nh\nu}{kT}}}{Z}$$

here Z is the partition function which can be written as –

$$Z = \sum_{n=0}^{\infty} e^{-nh\nu/kT}$$

Simplifying we get –

$$\langle E \rangle = \frac{h\nu}{e^{h\nu/kT} - 1}$$

Again, we know the number of oscillators per unit volume within the frequency range ν and ν+dν is given by

$$n_\nu d\nu = 8\pi\nu^2 d\nu / c^3$$

Hence the radiant energy density within the frequency range ν and ν+dν is given by

$$E_\nu d\nu = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1} d\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{kT}} - 1} d\nu$$

$$E_\nu d\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{kT}} - 1} d\nu$$

This relation is known as the Planck's radiation law for energy density of black body. Again using, $\nu = \frac{c}{\lambda}$; that is $d\nu = \left| -\frac{c}{\lambda^2} d\lambda \right|$ so the above equation can be written in terms of wavelength as –

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1} d\lambda$$

Here are the two forms of Planck's law. The equation accurately describes the experimental curve across the entire range of wavelengths.

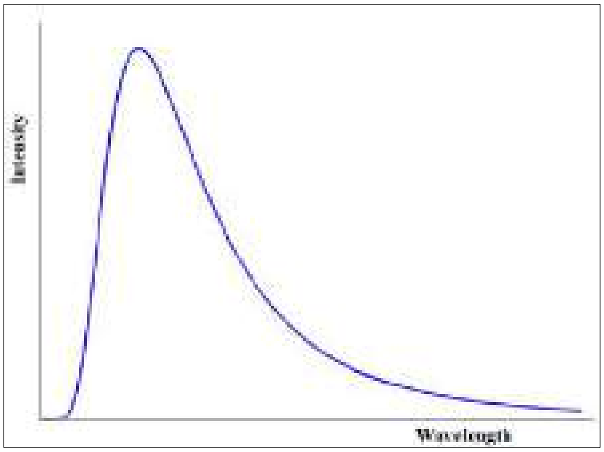


Fig: Planck’s distribution law

3.2.8. Application of Planck’s law:

From Planck's law, we can derive several important distributions of thermal radiation, including the Stefan-Boltzmann law, Wien's displacement law, and the Rayleigh-Jeans law.

Stefan-Boltzmann Law from Planck’s law

The Stefan-Boltzmann law states that the total energy radiated per unit surface area of a black body is proportional to the fourth power of its temperature.

To derive this law from the Planck’s law, the total energy radiated is obtained by integrating $E_\nu d\nu$ for waves of all frequencies from 0 to ∞ -

$$Q = \int_0^\infty E_\nu d\nu = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1} d\nu$$

Assuming, $x = \frac{h\nu}{kT}$

$$Q = \frac{8\pi k^4 T^4}{c^3 h^3}\int_0^\infty \frac{x^3}{e^x - 1} dx$$

$$Q = \frac{8\pi k^4 T^4}{c^3 h^3}\frac{\pi^4}{15} = \alpha T^4$$

This is the Stefan-Boltzmann law. Here the value of standard definite integral has been used –

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$$

The Stefan constant is given by –

$$\sigma = \frac{\alpha c}{4} = \frac{2k^4\pi^5}{15c^2h^3}$$

The value of Stefan constant is 5.67×10^{-8} W m^{-2} K^{-4}

Wien's radiation law from Planck's law

The Planck's radiation law is given by –

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}d\lambda$$

For short wavelength and low temperature, the term λT is very small so that the term $e^{\frac{hc}{\lambda kT}}$ of the denominator has considerable high value compared to 1. Thus the distribution reduces to –

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}e^{-\frac{hc}{\lambda kT}}d\lambda$$

This is same as the Wien's radiation law given by –

$$E_\lambda d\lambda = \frac{A}{\lambda^5}e^{-\frac{B}{\lambda T}}d\lambda$$

Where, $A = 8\pi hc$ and $B = \frac{hc}{k}$.

Rayleigh- Jeans law from Planck's law

The Rayleigh-Jeans law can be derived from Planck's law of black-body radiation by considering the limit where the wavelength is much longer (or the frequency is much lower) than the characteristic thermal wavelength (or frequency). The Planck's law is –

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}d\lambda$$

Thus for high value of wavelength and high temperature region the $\frac{hc}{\lambda kT}$ term is very small and the term $e^{\frac{hc}{\lambda kT}}$ can be approximated to –

$$e^{\frac{hc}{\lambda kT}} = 1 + \frac{hc}{\lambda kT}$$

Thus we have –

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}[\frac{1}{1+\frac{hc}{\lambda kT}-1}]$$

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{\lambda kT}{hc}d\lambda$$

$$E_\lambda d\lambda = \frac{8\pi kT}{\lambda^4}d\lambda$$

This derivation shows that the Rayleigh-Jeans law is an approximation of Planck's law valid for low frequencies (or long wavelengths).

Wien's Displacement law from Planck's law

The Planck's law is –

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}} - 1}d\lambda$$

For the emissive power to become maximum, the denominator must be minimum in the above equation. Let us write the denominator as –

$$f(\lambda) = \lambda^5(e^{\frac{hc}{\lambda kT}} - 1)$$

This function becomes minimum when $\frac{df}{d\lambda} = 0$

$$\frac{df}{d\lambda} = 5\lambda^4\left(e^{\frac{hc}{\lambda kT}} - 1\right) + \lambda^5 e^{\frac{hc}{\lambda kT}}\,(-\frac{hc}{kT\lambda^2})$$

To satisfy the condition for minimum value –

$$5\lambda^4\left(e^{\frac{hc}{\lambda kT}} - 1\right) + \lambda^5 e^{\frac{hc}{\lambda kT}}\left(-\frac{hc}{kT\lambda^2}\right) = 0$$

$$5\lambda^4 e^{\frac{hc}{\lambda kT}} - 5\lambda^4 - \frac{hc}{kT\lambda^2}\lambda^5 e^{\frac{hc}{\lambda kT}} = 0$$

$$1 - e^{-\frac{hc}{\lambda kT}} - \frac{hc}{5kT\lambda} = 0$$

$$1 - e^{-x} = \frac{x}{5}$$

Here we have assumed, $x = \frac{hc}{kT\lambda}$

The above equation cannot be solved directly as it is a transcendental equation. This type of equations are solved graphically. The point of intersection of the two graphs $(1 - e^{-x})$ and $\frac{x}{5}$ gives the solution which is here x = 4.9651

$$x = \frac{hc}{kT\lambda}$$

$$\lambda_m T = \frac{hc}{kx} = constant$$

This is the Wien's displacement law.

3.3. Photoelectric effect

The photoelectric effect is a phenomenon where electrons are emitted from a metallic surface due to absorption of electromagnetic radiation of a suitable frequency. The emitted electrons in this process are known as photoelectrons. The historical importance of the discovery of this phenomenon lies on the fact that photoelectric effect directly supports Planck's quantum hypothesis and provides evidence in favor of the quantum nature of light.

3.3.1 Experimental study of photoelectric effect

The experimental set up has been shown below –

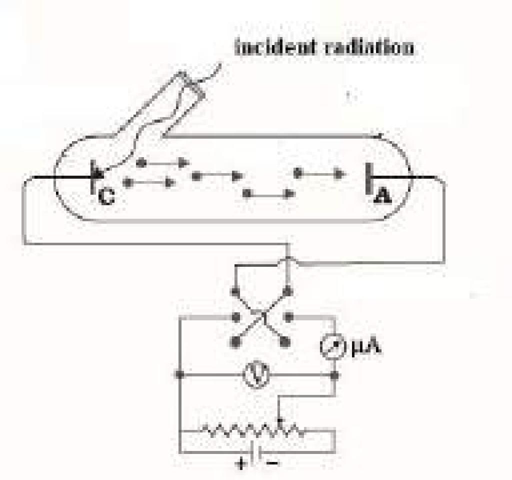


Fig: Experimental arrangement to study photoelectric effect

Schematic diagram of the experimental arrangement has been shown in the figure. The main component is an evacuated tube having a quartz window through which radiation can enter inside the tube. There are two metallic plates C and A placed inside the tube parallel to each other. Here the plate C is coated with some alkali metal and when a radiation is incident on the cathode C electrons get emitted from its surface. The electrode A is kept at a comparatively positive potential than C. Due to the potential difference the emitted photoelectrons get accelerated towards A and as a result photoelectric current flows through the circuit which is measured by an ammeter connected in the external circuit. The potential difference between C and A can be measured using a voltmeter.

The stopping potential: It has been observed that if the anode A is made negative potential with respect to C, the photoelectrons are retarded and as a result the photocurrent decreases with the increase of the potential difference and at a point the photocurrent completely stops.

The stopping potential refers to the potential difference between C and A at which the most energetic photoelectrons are just stopped from reaching the anode. At this voltage, the electric potential energy is equal to the maximum kinetic energy of the photoelectrons

$$eV_s = \frac{1}{2}mv_{max}^2$$

Here ‘e’ refers to the charge of the electron and V_s is the stopping potential.

3.3.2 Laws of photoelectric emission: By closely studying the experimental outcomes following laws has been derived for photoemission –

1. It has been observed that for a particular material, there is a minimum value of the frequency of the incident radiation below which no photoelectron emission takes place. This minimum value of the frequency is known as the threshold frequency (ν_0) of that material. Therefore, if we have a light of frequency ν, then it must satisfy $\nu \geq \nu_0$ for photoelectron emission to occur for a particular material.

2. The number of photoelectrons emitted per unit time, that is the photoelectric current is directly proportional to the intensity of the incident electromagnetic radiation. Intensity of light refers to the number of photons incident upon the metallic surface per unit time. It has also been observed that the kinetic energy of the emitted photoelectrons remains unaffected by the intensity of light.

3. The maximum kinetic energy of the emitted photoelectrons is directly proportional to the frequency of the incident light and is independent of its intensity. The kinetic energy (K.E.) of the photoelectrons is given by:

$$\frac{1}{2}mv_m^2 = h\nu - \varphi$$

where h is Planck’s constant, ν is the frequency of the incident light, and φ is the work function of the material which refers to the minimum energy which is required to release an electron from a metallic surface.

4. The emission of photoelectrons occurs almost instantaneously with the absorption of light ($< 10^{-8}$ s). There is no observable time lag between the absorption of a photon and the emission of an electron. This implies a direct and immediate interaction between the incident photon and the electron.

5. Experimentally it has been observed that the stopping potential is independent of the intensity of the incident radiation. The stopping potential increases with the frequency of the incident radiation.

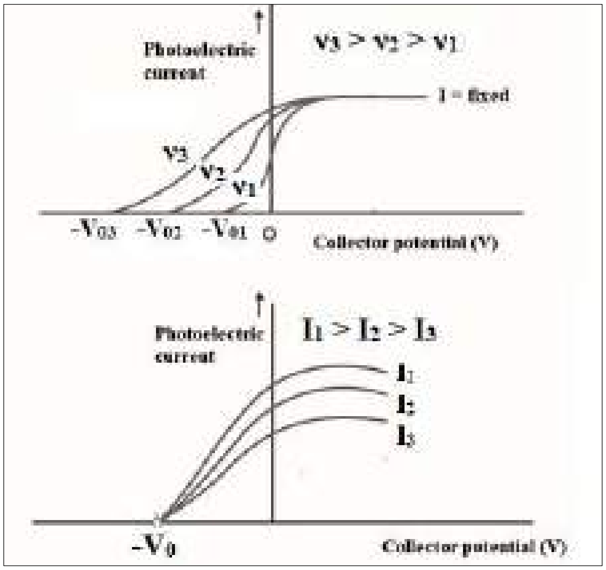


Fig: Variation of photoelectric current with collector potential at different frequencies and different intensities of the incident radiation

3.3.3. Key Failures of Classical Theory

1. The classical electromagnetic theory fails to explain the existence of the threshold frequency for any material. Classical theory predicts that the energy of the photoelectron is dependent upon the intensity of the incident radiation. So, given sufficient amount of intensity of the incident radiation can initiate photoelectric effect. However, it has been observed that the photoelectric effect is not possible below a certain frequency for a particular material irrespective of how the intense is the incident radiation.

2. The classical theory suggests that an electron in the metallic surface will require some time to accumulate sufficient amount of energy from the incident radiation for photoelectric emission. But the experimental study indicates that the photoelectric effect occurs instantaneously. There is no time lag between the incident of electromagnetic radiation and ejection of photoelectrons.

3. Classical theory predicts that the kinetic energy of emitted electrons is proportional to the intensity of the incident radiation. In reality, the kinetic energy of emitted electrons increases with the frequency of the incident light and is independent of light intensity.

$$\frac{1}{2}mv_m^2 = h\nu - \varphi$$

4. The photoelectric current has been found to be proportional to the light intensity, provided the frequency is above the threshold value for that material. The classical theory also fails to explain why photocurrent is intensity dependent.

3.3.4. Einstein’s Photoelectric equation

Albert Einstein provided a successful explanation of the photoelectric effect using quantum theory, which introduces the concept of photons –

1. Electromagnetic radiation is composed of discrete packet or quanta of energy called photon moving with the speed of light in space. The energy of a photon is given by, $E = h\nu$, where h is Planck's constant and ν is the frequency of light. Only photons with sufficient energy (above a certain frequency threshold) can eject electrons from the material.

2. The energy of an incident photon is such localized that it only can transfer its total energy to an electron in the metallic surface during collision, no fractional energy transfer is allowed in the process.

The maximum kinetic energy of a photoelectron is given by –

$$\frac{1}{2}mv_m^2 = h\nu - \varphi$$

The minimum energy required to eject an electron is called the work function (ϕ) of the material. When the total energy of the photon is used in releasing the photoelectron from the metallic surface we can write –

$$h\nu_0 - \varphi = 0$$

$$h\nu_0 = \varphi$$

Using this in the previous equation we get –

$$\frac{1}{2}mv_m^2 = h\nu - h\nu_0 = h(\nu - \nu_0)$$

This is another form of Einstein’s photoelectric equation. His observations clearly suggests that –

i. The maximum value of the kinetic energy of the photoelectron depends upon the frequency of the incident radiation.

ii. This equation supports the presence of threshold frequency below which photoelectric effect is not possible.

iii. Increase of the intensity of light increases the photocurrent, the equation suggests that the maximum energy of the photoelectron does not depend on intensity.

iv. Electrons absorb energy from individual photons instantaneously.

3.4. Compton effect

When a photon collides with the a free or loosely bound electron of the scatterer at rest, it transfers some energy to the electron. Therefore, the scattered photon will have a smaller energy and longer wavelength compared to the incident photon. The modification of the wavelength of scattered photon by considering the elastic collisions between the incident photon and the free electron of the scattered material is known as Compton effect.

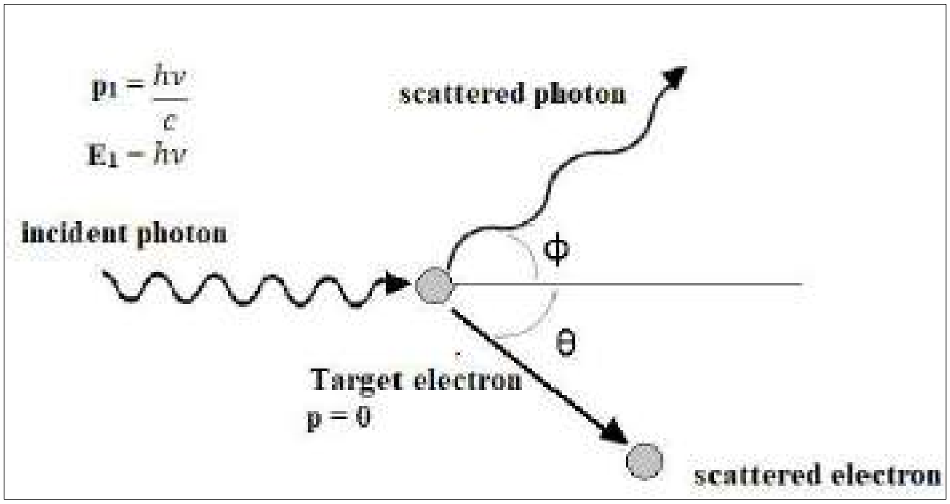


Fig: Schematic representation of Compton effect

Let a photon having energy hν collides with an electron of the scatterer at rest. Due to the collision, the photon scatters at an angle φ and the electron recoils at an angle θ. The scattering process has been shown schematically in the above diagram. ν' is the frequency of the scattered photon. The laws of conservation of energy and momentum have been used to measure the modification in wavelength of the scattered photon.

Applying the law of conservation of momentum:

Along the direction of incidence of photon–

$$\frac{h\nu}{c} + 0 = \frac{h\nu'}{c}\cos\varphi + p\cos\theta \qquad \text{(i)}$$

Along the perpendicular direction of incidence of photon –

$$0 = \frac{h\nu'}{c}\sin\varphi - p\sin\theta \qquad \text{(ii)}$$

p represents the momentum of the recoil electron. From the equation (i) and (ii) we have –

$$pc\cos\theta = h\nu - h\nu'\cos\varphi \qquad \text{(iii)}$$

$$pc\sin\theta = h\nu'\sin\varphi \qquad \text{(iv)}$$

Squaring and adding (iii) and (iv) we get –

$$(pc)^2 = (h\nu)^2 - 2(h\nu)(h\nu')\cos\varphi + (h\nu')^2 \qquad \text{(v)}$$

Applying the law of conservation of energy we get the energy of the recoil electron –

$$E^e = kinetic\ energy\ of\ the\ recoil\ electron + its\ rest\ mass\ energy$$

$$E^e = energy\ transferred\ to\ electron\ by\ photon + its\ rest\ mass\ energy$$

$$E^e = (h\nu - h\nu') + m_0c^2 \qquad \text{(vi)}$$

Again, for a relativistic recoil electron the energy is given by –

$$E^e = \sqrt{p^2c^2 + m_0{}^2c^4} \qquad \text{(vii)}$$

Hence, we get from equation (vi) and (vii)-

$$\sqrt{p^2c^2 + m_0{}^2c^4} = (h\nu - h\nu') + m_0c^2$$

$$p^2c^2 + m_0{}^2c^4 = [(h\nu - h\nu') + m_0c^2]^2$$

$$p^2c^2 + m_0{}^2c^4 = (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu') + 2m_0c^2(h\nu - h\nu') + m_0{}^2c^4$$

$$p^2c^2 = (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu') + 2m_0c^2(h\nu - h\nu') \qquad \text{(viii)}$$

Thus, from equation (v) and (viii) we get –

$$2m_0c^2(h\nu - h\nu') = 2(h\nu)(h\nu') - 2(h\nu)(h\nu')\cos\varphi$$

$$2h(\nu - \nu')m_0c^2 = 2h^2\nu\nu'(1 - \cos\varphi)$$

$$\frac{(\nu - \nu')}{\nu\nu'} = \frac{h}{m_0c^2}(1 - \cos\varphi)$$

$$\frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{m_0c^2}(1 - \cos\varphi)$$

$$\frac{1}{c/\lambda'} - \frac{1}{c/\lambda} = \frac{h}{m_0c}(1 - \cos\varphi)$$

$$\boldsymbol{\lambda' - \lambda = \frac{h}{m_0c}(1 - \cos\varphi)}$$

This is the expression of Compton shift of the scattered photon by the electron. The equation suggests that the Compton shift is a function of scattering angle and does not depend upon the wavelength of the incident photon. The Compton shift is always positive as the maximum calue of $\cos\varphi$ is 1.The quantity $\frac{h}{m_0c}$ is known as Compton wavelength (λ_C) of the scattering particle.

The Compton wavelength (λ_C) is given by –

$$\boldsymbol{\lambda_C = \frac{h}{m_0c}}$$

The Compton shift is given by –

$$\boldsymbol{\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\varphi)}$$

When, $\varphi = \pi$ we get –

$$\boldsymbol{\lambda' - \lambda = \frac{2h}{m_0 c}}$$

Therefore the Compton shift becomes maximum at $\varphi = \pi$

For an electron the Compton wavelength (λ_C) is given by –

$$\lambda_C = \frac{h}{m_0 c}$$

Here, $m_0 = 9.1\times10^{-31}$ kg is the rest mass of an electron.

$$\lambda_C = \frac{6.626 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^{8}} = 0.2427\ \text{Å}$$

The Compton effect, or Compton scattering, involves the scattering of X-ray or gamma-ray photons by electrons. When a photon collides with an electron, it transfers some of its energy to the electron, causing the electron to be ejected and the photon to be scattered with reduced energy. The phenomenon demonstrates the particle nature of light and supports the concept of photons.

3.5 Wave-Particle Duality: de Broglie hypothesis

Wave-particle duality can be regarded as one of the most fundamental concepts of quantum mechanics. Wave-particle duality highlights the simultaneous exhibition of both wave and particle type characteristics of an entity. This principle has been verified through various experiments. As we have discussed previously blackbody radiation, photoelectric effect with the help of quantum theory of light which suggests that the light is a stream of photon particles moving with velocity of light in space. At the same time the concept of photon theory failed to explain the phenomena of diffraction and interference which requires wave theory of light for complete understanding. Therefore, we can say that electromagnetic radiations sometime act as wave and sometime like photon particle. Similarly, one can expect that the particles will show wave like properties in some cases.

de Broglie hypothesis

Louis de Broglie proposed that any moving particle or object has a wave associated with it, with the wavelength (λ) given by:

$$\lambda = h / p$$

where h is Planck’s constant and p is the momentum of the particle. Electrons and other particles also exhibit wave-like properties. This has been confirmed by electron diffraction experiments, where electrons produce interference patterns when passed through a crystal.

From the Planck’s quantum theory, we know $E = h\nu$

Again for a relativistic particle we also know $E = \sqrt{p^2c^2 + {m_0}^2c^4}$

Again, the rest mass (m_0) of photon is zero, which suggests that $E = pc$

From the two relation we can write –

$$h\nu = pc$$

$$p = \frac{h\nu}{c}$$

$$\boldsymbol{p = \frac{h}{\lambda}}$$

Here we have used the relation $c = \nu\lambda$

3.5.1 Expression of the de-Broglie wavelength of an electron moving inside a potential difference V:

Let us consider an electron of charge ‘e’ and mass ‘m’ has been subjected to a potential difference ‘V’ so that it moves with a velocity ‘v’, then the kinetic energy of the electron is –

$$E = \frac{1}{2}mv^2$$

Here,

$$eV = \frac{1}{2}mv^2$$

$$v = \sqrt{\frac{2eV}{m}} = \sqrt{\frac{2E}{m}}$$

If the de Broglie wavelength of this moving electron of momentum ‘p’ is λ then –

$$\lambda = \frac{h}{p}$$

$$\lambda = \frac{h}{mv}$$

After substituting the value of velocity obtained above we get –

$$\lambda = \frac{h}{m\sqrt{\frac{2eV}{m}}}$$

$$\lambda = \frac{h}{\sqrt{2meV}}$$

Which by using eV = E, can also be written as –

$$\lambda = \frac{h}{\sqrt{2mE}}$$

3.6 Phase and Group Velocity:

3.6.1 Phase velocity

A progressive wave of amplitude 'a' propagating with velocity 'v' along the positive x direction can be written as –

$$y = a \sin \omega (t - \frac{x}{v})$$

Here, ω represents angular frequency and the wave velocity 'v' also known as phase velocity (v_p). From the above equation the phase (φ) of the wave at any instant of time is –

$$\varphi(x, t\,) = \omega (t - \frac{x}{v})$$

Differentiating with respect to time we get –

$$\frac{\partial \varphi}{\partial t} = \omega \left(1 - \frac{1}{v}\frac{dx}{dt}\right)$$

For a point with constant phase, we have –

$$\frac{\partial \varphi}{\partial t} = 0$$

$$v = (\frac{dx}{dt})_{\varphi}$$

The velocity with which a wave having constant phase moves forward is called phase velocity (v_p). The velocity with which a definite phase of a wave advances in a medium is called phase velocity or wave velocity.

$$y = a \sin \omega \left(t - \frac{x}{v_p}\right) = a \sin \left(\omega t - \omega \frac{x}{v_p}\right) = a \sin(\omega t - kx)$$

Here 'k' is the propagation constant or phase constant of the wave.

$$k = \frac{\omega}{v_p}$$

Thus phase velocity is given by $v_p = \frac{\omega}{k}$

3.6.2 Group Velocity:

The superposition of multiple number of waves will form a complex wave called a wave packet. Inside the wave packet the different constituent waves move with different phase velocities but the envelop comprising the wave group advances through the medium with a constant velocity which is called the group velocity.

Let us consider a wave group arises from the combination of two harmonic waves with slightly differing angular frequency Δω and wave number Δk.

Two interfering waves are given by –

$$y_1 = a\sin(\omega t - kx)$$

$$y_2 = a\sin[(\omega + \Delta\omega)t - (k + \Delta k)x]$$

The resultant displacement at any time is given by –

$$y = y_1 + y_2$$

$$y = a\sin(\omega t - kx) + a\sin[(\omega + \Delta\omega)t - (k + \Delta k)x]$$

$$y = 2a\cos(-\frac{\Delta\omega}{2}t + \frac{\Delta k}{2}x)\sin[\left(\frac{2\omega + \Delta\omega}{2}\right)t - \left(\frac{2k + \Delta k}{2}\right)x]$$

Since, Δω and Δk are small compared to ω and k, we can neglect Δω and Δk in the sine terms.

$$y = 2a\cos(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x)\sin(\omega t - kx)$$

This equation shows that the resultant wave is an amplitude modulated wave which is travelling with phase velocity

$$v_p = \frac{\omega}{k}$$

Again, from the above equation it is clear that the amplitude of the resultant group of waves varies with time with circular frequency $\frac{\Delta\omega}{2}$ and it is given by –

$$A = 2a\cos(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x)$$

The variation of this amplitude produces successive wave group.

Group velocity is the velocity with which the maximum amplitude of the resultant group of waves moves.

At x = 0 and t = 0, the maximum amplitude is

$$A_{max} = 2a$$

The group velocity is given by –

$$v_g = \lim_{\Delta k \to 0} \frac{\frac{\Delta\omega}{2}}{\frac{\Delta k}{2}} = \lim_{\Delta k \to 0} \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

$$v_g = \frac{d\omega}{dk}$$

3.7 Heisenberg's uncertainty principle

In case of Classical mechanics, we can simultaneously measure the position and momentum of a moving particle at any instant with perfect accuracy. However, for quantum mechanics a moving particle is associated with a wave packet and the particle may be found anywhere within the wave packet. Hence, there will be an uncertainty in the specification of the position of a particle. At the same time a wave packet also consist of a range of wavelength and according to the de Broglie relation there will be also an uncertainty in the measurement of momentum of a particle.

Heisenberg a German scientist in 1927, gave uncertainty principle which states that "The determination of exact position and momentum of a moving particle simultaneously at any instant of time is impossible''.

In general, if $\triangle$x represents the uncertainty in measurement of position of particle along x-axis, and $\triangle$p represents the uncertainty in the measurement of momentum, then the position – momentum Heisenberg uncertainty relation gives –

$$\Delta \boldsymbol{x} . \Delta \boldsymbol{p}_{\boldsymbol{x}} \geq \hbar/\mathbf{2}$$

3.8. Wave function

The space-time behavior of quantum mechanical particle is governed by a function which is known as wave function. Wave function is generally denoted by ψ (**r**, t). This is related to the probability of finding a particle about a position at a particular time.

The probability density of a particle that is the probability of founding the particle per unit volume at a given space at a particular time. It is given by - $\psi(r,t)^* \, \psi(r,t) = |\psi(r,t)|^2$

Therefore, the probability of obtaining a particle inside a volume element dτ at some particular time t is given by = $|\psi(r,t)|^2 \, d\tau$

As the particle must be located somewhere in space, we can write –

$$\int_{-\infty}^{+\infty} |\psi(r,t)|^2 \, d\tau = 1$$

$$\int_{-\infty}^{+\infty} \psi(r,t)^* \, \psi(r,t) \, d\tau = 1$$

This is termed as the normalization condition of the wave function.

Characteristics of wave function are –

(i) The wave function must be finite everywhere.

(ii) The wave function must be single valued.

(iii) The wave function must be continuous and have a continuous first order derivative.

(iv)The wave function should be a square integrable function.

(v) The magnitude of ψ (**r**, t) is large in the regions where the probability of finding the particle is high and is small in the regions where the probability of finding it is low.

3.9. Basic postulates of quantum mechanics

Postulate 1: The state of a quantum mechanical system is completely specified by the function ψ (**r**, t) that depends on the coordinates of the particle, **r** and the time t. This function is called the wavefunction or state function and has the property that $\psi(r,t)^* \, \psi(r,t)d\tau$ is the probability that the particle lies in the volume element $d\tau$ located at r and time t.

This is the probabilistic interpretation of the wavefunction. As a result the wavefunction must satisfy the condition that finding the particle somewhere in space is 1 and this gives us the normalization condition

$$\int_{-\infty}^{+\infty} \psi(r,t)^* \, \psi(r,t)d\tau \;= 1$$

The other conditions on the wavefunction that arise from the probabilistic interpretation are that it must be single valued, continuous and finite. We normally write wavefunctions with a normalization constant included.

Postulate 2: To every observable in classical mechanics there corresponds a linear Hermitian operator in quantum mechanics. This postulate comes from the observation that the expectation value of an operator that corresponds to an observable must be real and therefore the operator must be Hermitian. Some examples of Hermitian operators are –

The expression of the position operator $(\widehat{p_x}) \;=\; \frac{\hbar}{i}\frac{\partial}{\partial x}$

The expression of the kinetic energy operator $(\hat{T}) = -\frac{\hbar^2}{2m}\frac{\delta^2}{\partial x^2}$

Postulate 3: In any measurement of the observable associated with operator $\boldsymbol{\hat{A}}$, the only values that will ever be observed are the eigenvalues, $\boldsymbol{a}$, that satisfy the eigenvalue equation-

$$\hat{\boldsymbol{A}}\boldsymbol{\psi} = \boldsymbol{a}\boldsymbol{\psi}$$

This is the postulate that the values of dynamical variables are quantized in quantum mechanics (although it is possible to have a continuum of eigenvalues in the case of unbound state). If the system is in an eigenstate of $\hat{\boldsymbol{A}}$ with eigenvalue $\boldsymbol{a}$ then any measurement of the quantity $\hat{\boldsymbol{A}}$ will always yield the value $\boldsymbol{a}$.

Postulate 4: If a system is in a state described by the normalised wavefunction, $\boldsymbol{\psi}$**,** then the average value of the observable corresponding to $\hat{\boldsymbol{A}}$ is given by –

$$< \hat{\boldsymbol{A}} >= \int_{-\infty}^{+\infty} \psi^* \hat{\boldsymbol{A}} \ \psi \, d\tau$$

Postulate 5: The wavefunction or state function of a system evolves in time according to the time-dependent Schrödinger equation –

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t) = i\hbar\frac{\partial \Psi(x,t)}{\partial t}$$

3.10 Operator in Quantum Mechanics

An operator can be defined as a mathematical rule that changes a given function into a new function. In quantum mechanics each dynamical variable (position, momentum, energy etc.) is represented by a linear operator. If **A** is an operator, it is represented by $\hat{\boldsymbol{A}}$**.**

Position operator

Suppose, the wave function for a free particle moving in the positive x-direction is

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

$\therefore \quad \frac{\partial \psi}{\partial x} = ikAe^{i(kx-\omega t)}$

Again, de Broglie equation of matter wave gives –

$$p = \frac{h}{\lambda} = \frac{h}{2\pi}.\frac{2\pi}{\lambda} = \hbar k$$

Therefore, we have,

$$\frac{\partial \psi}{\partial x} = ikAe^{i(kx-\omega t)}$$

$$\frac{\partial \psi}{\partial x} = ik\psi$$

$$\frac{\hbar}{i}\frac{\partial \psi}{\partial x} = \hbar k \ \psi$$

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial x} = p_x\,\psi$$

Therefore, the expression of the momentum operator (p_x): $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$

Energy operator.

Suppose, the wave function for a free particle in the positive x-direction is

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

$\therefore \quad \frac{\partial\psi}{\partial t} = -i\omega Ae^{i(kx-\omega t)}$

$\therefore \quad \frac{\partial\psi}{\partial t} = -i\omega\psi$

$$i\hbar\frac{\partial\psi}{\partial t} = \hbar\omega\psi$$

From Planck's law we know: $E = \hbar\omega$

Therefore we have,

$$i\hbar\frac{\partial\psi}{\partial t} = E\psi$$

Therefore, the expression of the energy operator: $\hat{E} = i\hbar\frac{\partial}{\partial t}$

Commutation relation: Commutator of two arbitrary operators A and B is defined as –

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

If $[\hat{A},\hat{B}] = 0$, then $\hat{A}\hat{B} = \hat{B}\hat{A}$, then the operators are called commutative.

If $[\hat{A},\hat{B}] \neq 0$, then $\hat{A}\hat{B} \neq \hat{B}\hat{A}$, then the operators are called non-commutative.

The commutation relation between position and momentum operator $[\hat{x},\widehat{p_x}]$:

$$[\hat{x},\widehat{p_x}]f(x) = [\hat{x}\widehat{p_x} - \widehat{p_x}\hat{x}]f(x)$$

$$= x\frac{\hbar}{i}\frac{\partial}{\partial x}f(x) - \frac{\hbar}{i}\frac{\partial}{\partial x}(xf(x))$$

$$= x\frac{\hbar}{i}\frac{\partial}{\partial x}f(x) - x\frac{\hbar}{i}\frac{\partial}{\partial x}f(x) - \frac{\hbar}{i}f(x)$$

$$[\hat{x},\widehat{p_x}]f(x) = -\frac{\hbar}{i}f(x)$$

$$[\hat{x},\widehat{p_x}] = i\hbar$$

The relation suggests that the position and momentum operators do not commute which further indicates that it is impossible to measure the position and momentum of a particle simultaneously with same accuracy.

3.11 Schrödinger equation

The Schrodinger equation in one dimension is given by –

$$-\frac{ħ^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2}+V(x)\Psi(x,t)=iħ\frac{\partial\Psi(x,t)}{\partial t}$$

Here, ψ (x,t) is the wave function, V (x) is the potential and m is the mass of the particle under consideration.

We have to solve this equation to obtain the time independent Schrodinger equation.

It has been assumed that V is independent of time. In that case the Schrödinger equation can be solved by the method of separation of variables by using –

$$\Psi(x,t)=\ \psi(x)\ \varphi(t)$$

The lower case ψ(x) is a function of only position and φ (t) is a function of only time.

For separable solutions we can write –

$$\frac{\partial\Psi}{\partial t}=\psi\frac{d\varphi}{dt}$$

$$\frac{\partial^2\Psi}{\partial x^2}=\varphi\frac{d^2\psi}{dx^2}$$

Substituting these into the Schrodinger equation we get –

$$-\frac{ħ^2}{2m}\varphi\frac{d^2\psi}{dx^2}+V(x)\psi(x)\ \varphi(t)=iħ\psi\frac{d\varphi}{dt}$$

Dividing both side of the equation by φψ we get –

$$-\frac{ħ^2}{2m\psi}\frac{d^2\psi}{dx^2}+V(x)=\frac{iħ}{\varphi}\frac{d\varphi}{dt}$$

Now, the left side is a function of t alone, and the right side is a function of x alone. The only way this can possibly be true is if both sides are in fact constant—otherwise, by varying t, one could change the left side without touching the right side, and the two would no longer be equal. Assuming the separation constant as 'E' which is the total energy of the system we have –

$$\frac{iħ}{\varphi}\frac{d\varphi}{dt}=E$$

$$\frac{d\varphi}{dt}=-\frac{iE}{ħ}\varphi$$

On the other hand,

$$-\frac{ħ^2}{2m\psi}\frac{d^2\psi}{dx^2}+V(x)=E$$

$$-\frac{ħ^2}{2m}\frac{d^2\psi}{dx^2}+V(x)\psi=E\psi$$

separation of variables has turned a partial differential equation into two ordinary differential equations. The first of these is easy to solve (just multiply through by dt and integrate); the general solution is,

$$\varphi(t)\sim e^{-\frac{iEt}{ħ}}$$

The second is called the time-independent Schrödinger equation; we can solve this equation if the potential is specified.

Stationary state:

The complete wave function is given by –

$$\Psi(x,t)=\psi(x)\,\varphi(t)$$

The probability density is given by –

$$|\Psi(x,t)|^2=\Psi^*\Psi=\psi^*(x)e^{+\frac{iEt}{ħ}}\,\psi(x)e^{-\frac{iEt}{ħ}}=|\Psi(x)|^2$$

It does not depend upon time and the same thing happens in calculating the expectation

value of any dynamical variable. Every expectation value is constant in time. These states are called stationary states.

Superposition principle

According to this principle, if the Schrödinger equation has multiple number of solutions like ψ_1, ψ_2, ψ_3….ψ_n then their linear combination will also be a solution of the Schrödinger equation.

$$\psi=\ c_1\psi_1+c_2\psi_2+c_3\psi_3\ldots+c_n\psi_n=\sum_n c_n\psi_n$$

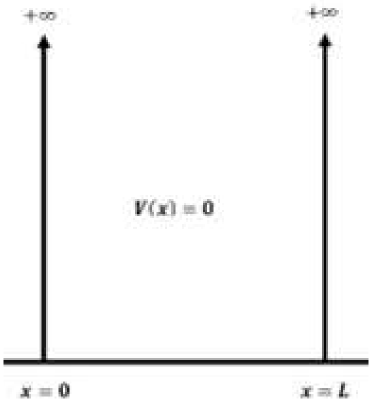
Here, c_n coefficient contains the information of probability of occupying state ψ_n. The probability of the particle being at state ψ_n is $|c_n|^2$

3.12 Schrödinger equation for a particle moving freely inside a one dimensional infinitely deep potential well of width ‘L’

The time independent Schrödinger equation is given by:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x)=E\psi(x)$$

Ψ(x) is the wave function, V(x) is the potential energy and E is the total energy of the system.



For a particle moving inside an one dimensional potential box the condition is:

$$V(x) = 0 \text{ at } 0 < x < L$$

$$= \infty \text{ otherwise}$$

As the particle is moving freely inside the potential box and it cannot exit outside the box, therefore the Schrödinger equation for this region (V(x) = 0) will be:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

$$\frac{d^2\psi(x)}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x)$$

Let us assume that, $\frac{2mE}{\hbar^2} = k^2$ and substituting in the above equation we get:

$\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0$ ------(1)

The solution of this equation is well known and can be written as:

$\psi(x) = A\sin kx + B\cos kx$ ------(2)

Here, A and B are the constants whose value can be obtained by using the boundary condition:

(i) Ψ(x) = 0 at x = 0

(ii) Ψ(x) = 0 at x = L

The first boundary condition yields: $A\sin 0 + B\cos 0 = 0$ which gives B = 0

Thus the equation (2) reduces to: $\psi(x) = A\sin kx$ ------- (3)

Now applying the second boundary condition to equation (3) we get: $A\sin kL = 0$

As 'A' cannot be zero, we therefore have, $\sin kL = 0$ which gives

$$kL = n\pi$$

$$\therefore k = \frac{n\pi}{L} \text{ -----(4)}$$

Therefore the equation (3) can be written as: $\psi_n(x) = A \sin\frac{n\pi x}{L}$ ----(5)

Again we know that it is certain that the particle is moving inside the box and the normalization condition is:

$$\int_0^L \psi_n^*(x)\psi_n(x)dx = 1$$

Using the value of the wave function from the equation (5) we have:

$$|A|^2 \int_0^L sin^2\frac{n\pi x}{L}dx = 1$$

Integrating and putting the limiting values we obtain: $A = \sqrt{\frac{2}{L}}$

Therefore the eigenfunction of the system is given by: $\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$

Initially we have assumed that $\frac{2mE}{\hbar^2} = k^2$ and we have obtained $k = \frac{n\pi}{L}$.

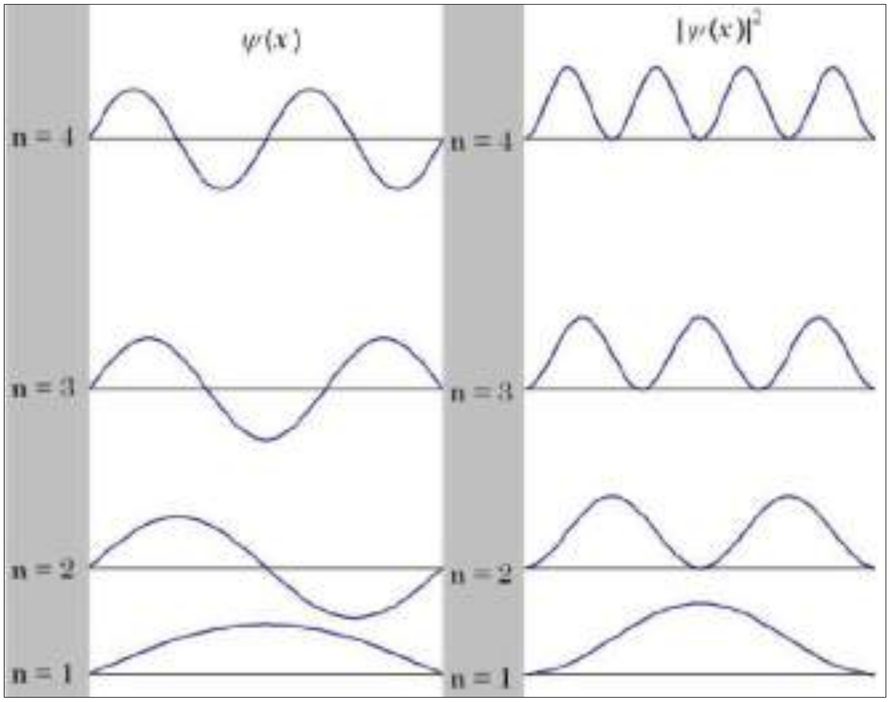
Equating these two relations we can easily obtain the value of eigen energy of the system:

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

Which can also be written as:

$$E_n = \frac{n^2h^2}{8mL^2}$$

The plot of wave function of first few states of a particle in a box –



EXERCISE

1 Multiple Choice Type Questions

(i)**. The wave function $\boldsymbol{\psi_m}(\boldsymbol{x})$ and $\boldsymbol{\psi_n}(\boldsymbol{x})$ are orthogonal to each other, then the correct option is:**

(a) $\int_{-\infty}^{+\infty} \psi_m(x)^* \psi_n(x)dx = 1$ (b) $\int_{-\infty}^{+\infty} \psi_m(x)^* \psi_n(x)dx = 0$

(c) $\int_{-\infty}^{+\infty} \psi_m(x)^* \psi_n(x)dx = 1/2$ (d) $\int_{-\infty}^{+\infty} \psi_m(x) \psi_n(x)dx = 1$

(ii) **The expression of the momentum operator is given by:**

(a) $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ (b) $\hat{p} = \frac{h}{2\pi i}\frac{\partial}{\partial t}$ (c) $\hat{p} = \frac{h}{\pi}\frac{\partial}{\partial x}$ (d) $\hat{p} = \frac{1}{2\pi i}\frac{\partial}{\partial x}$

(iii) **The eigenvalue of the operator $\boldsymbol{\frac{d^2}{dx^2}}$ for the eigenfunction $\boldsymbol{e^{ix}}$ is:**

(a) 4 (b) -1 (c) 5 (d) 0.5

(iv) **The normalized ground state wave function of a particle inside a one dimensional potential box of length 'L' is given by:**

(a) $\psi(x) = \sqrt{\frac{2}{L}}\sin\frac{L}{n\pi x}$ (b) $\psi(x) = \sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}$ (c) $\psi(x) = \sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}$

(d) $\psi(x) = \sqrt{\frac{2}{L}}\sin\frac{\pi x}{L^2}$

(v) **The ground state energy of a particle moving in a one dimensional potential box is given in terms of length 'l' of the box by:**

(a) $\frac{2h^2}{8ml^2}$ (b) $\frac{h^2}{8ml^2}$ (c) $\frac{h}{8ml^2}$ (d) 0.

Ans: (b) $\frac{h^2}{8ml^2}$

(vi) **If the width of a potential well is increased to twice of its initial width then the ground state energy of the system:**

(a) becomes 1/4th of its initial value. (b) becomes ½ of its initial value.

(c) becomes 1/6th of its initial value. (d) remains same.

(vii) **In Stefan's law, the total energy emitted per unit area per second from perfectly black body is proportional to-**

(a) T^2 (b) T^3 (c) T^4 (d) T^5

(viii) **The expression of the Compton wavelength is given by -**

(a) $\lambda = \frac{h}{m_0c^2}$ (b) $\lambda = \frac{h}{m_0}$ (c) $\lambda = \frac{h}{m_0c}$ (d) $\lambda = \frac{h}{c^2}$

(ix) **The rest mass of photon is –**

(a) 0 (b) its relativistic mass (c) undefined (d) none of these

(x) **The absorptive power of a black body is –**

(a) 1 (b) 0 (c) 2 (d) ∞

2. Numerical problems:

(i) An electron of mass 9.11×10-31 kg is moving in one dimensional infinitely deep potential well of width 1 Å. Find (i) the ground state energy, (ii) energy of the 1st excited state and (iii) the frequency of emitted radiation due to transition of the electron from the 1st excited to ground state.

(ii) Compute the smallest uncertainty in the position of an electron moving with velocity 3×107 m/s. The rest mass of an electron is 9.11×10-31 kg.

(iii) A photon has energy 85 eV. Find its (a) frequency, (b) wavelength, (c) momentum, (d) mass and (e) number of photons required to produce 1 joule of energy.

(iv) A proton is moving freely with kinetic energy 43.9 eV. Calculate its de Broglie wavelength.

(v) What is the de Broglie wavelength of a thermal electron at 400 K?

(vi) A system has two energy states ε_0 and $3\varepsilon_0$. ψ_1 and ψ_2 are the corresponding normalized wave functions. At an instant the system is in a superpose state

$$\psi = C_1\psi_1 + C_2\psi_2$$

And given that $C_1 = \frac{1}{\sqrt{2}}$.

Find the value of C_2 for which ψ is normalized. What is the probability that an energy measurement would yield a value $3\varepsilon_0$? Find out the expectation value of energy.

(vii) The ground state and the excited state normalized wave functions of an atom are ψ0 and ψ1 respectively, the corresponding energy being E0 and E1. If the probability of finding the atom in the ground state is 90% and that for the excited state is 10%, then find the average energy of the atom. Also determine the normalized wave function.

(viii) Evaluate the expectation value of position for a one-dimensional potential box of length L in the ground state.

CHAPTER 4:

OSCILLATIONS AND VIBRATIONS

Chapter – 4

4.1. Periodic motion:

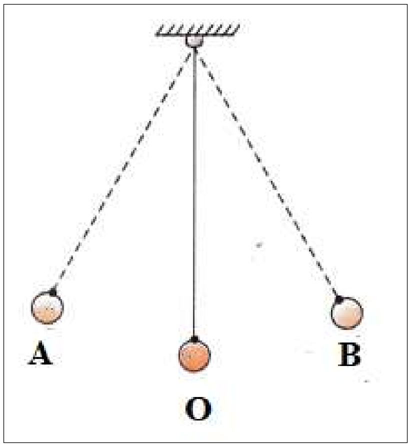
The motion which repeats itself at regular interval of time is called periodic motion. In this type of motion, the position, velocity, and acceleration of the object under motion follow a repetitive cycle. Some common examples of periodic motion are –

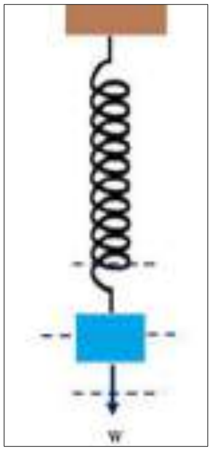
i. Motion of a simple pendulum

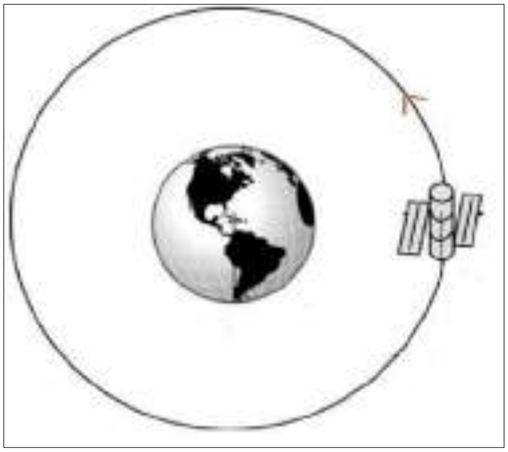
ii. The motion of a mass attached to a spring.

iii. The motion of an object in a circular path with uniform velocity like a satellite orbiting earth.

iv. Various type of waves like sound and light waves repeat their pattern at regular interval of space and time.







4.2 Oscillatory Motion

Oscillatory motion is a type of periodic motion where an object moves back and forth about an equilibrium position. Oscillatory motion specifically involves a repetitive to-and-fro motion.

Some examples of oscillatory motion are –

i. A mass attached to a spring,

ii. Motion of the bob of a simple pendulum,

iii. Vibrations of a tuning fork.

4.3. Periodic, Harmonic and Non-harmonic function:

Periodic Function A periodic function is any function that repeats its values at regular intervals or periods. Example: $\sin x$. Its period is 2π. The function repeats its value after every 2π unit.

Harmonic Function A harmonic function refers to a function that can be represented as a sinusoidal wave or a combination of sinusoidal waves. These functions are a subset of periodic functions. Example: $\sin x + \cos 2x$. This function is a sum of sinusoidal functions, making it harmonic.

Non-Harmonic Periodic Function A non-harmonic periodic function is a periodic function that cannot be expressed as a sum of simple sine and cosine functions. The square wave is a classic example of a non-harmonic periodic function.

4.4. Some important parameters related to oscillatory motion:

One complete oscillation: One complete oscillation refers to the motion of an object as it travels through a cycle that starts from a specific point, moves through all its possible positions, and then returns to the starting point.

A particle starting its motion from point A and then following AOB and BOA path to return at point A is known as one complete oscillation of the particle.

Time period (T): Time required for one complete oscillation is called time period. The unit of time period is the unit of time like second.

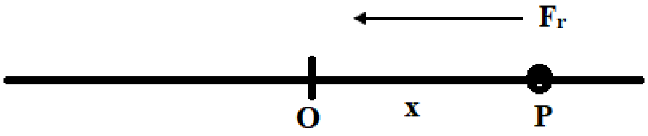
Frequency (n): Number of oscillation by a particle per unit time is known as frequency of the particle and its unit is Hz or s^{-1}.

Relation between frequency (n) and time period (T):

$$n = \frac{1}{T}$$

4.5 Simple Harmonic Motion:

This is a type of periodic motion where the object oscillates back and forth about some equilibrium position. This type of motion always involves a restoring force which is proportional to the distance of the particle from the equilibrium position and is oppositely directed to the displacement. The displacement in case of a simple harmonic motion with respect to time can be suitably represented by a sinusoidal graph.



In the figure a particle is at a distance ‘x’ from the origin O and a restoring force F_r is acting on the particle. This force tries to bring the particle back to its equilibrium position and the force is directly proportional to the distance from the equilibrium point.

According to the definition of the restoring force –

$$F_r = -ax$$

‘a’ is the force constant which has the unit N/m or dyne/cm in SI and CGS system, respectively.

4.5.1 Equation of motion: According to the Newton’s law:

$$Force = m\frac{d^2x}{dt^2}$$

Therefore, we can write –

$$m\frac{d^2x}{dt^2} = -ax$$

$$\frac{d^2x}{dt^2} = -\frac{a}{m}x$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$\omega = \sqrt{\frac{a}{m}}$ is the angular frequency of the simple harmonic oscillator. This is the differential equation of a body of mass ‘m’ executing simple harmonic motion. It is a second order differential equation.

4.5.2 Solution of the differential equation:

The differential equation of SHM is given by –

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

To solve the above equation, we shall follow a trick by multiplying the equation with $2\frac{dx}{dt}$ –

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} + 2\frac{dx}{dt}\omega^2 x = 0$$

By rearranging the above equation, we get –

$$\frac{d}{dt}(\frac{dx}{dt})^2 + \omega^2\frac{d}{dt}(x)^2 = 0$$

$$\frac{d}{dt}[\left(\frac{dx}{dt}\right)^2 + \omega^2 x^2] = 0$$

Now integrating this equation, we get –

$$\left(\frac{dx}{dt}\right)^2 + \omega^2 x^2 = Constant\ (say\ M)$$

Using a boundary condition, the velocity is zero ($\frac{dx}{dt} = 0$) at the maximum displacement $x = A$ from the equilibrium position we get $M = \omega^2 A^2$.

$$\left(\frac{dx}{dt}\right)^2 + \omega^2 x^2 = \omega^2 A^2$$

$$\left(\frac{dx}{dt}\right)^2 = \omega^2 (A^2 - x^2)$$

$$\frac{dx}{dt} = \omega\sqrt{A^2 - x^2}$$

By separating the variables and integrating we get –

$$\frac{dx}{\sqrt{A^2 - x^2}} = \omega\ dt$$

$$\int \frac{dx}{\sqrt{A^2 - x^2}} = \omega \int dt$$

$$sin^{-1}\frac{x}{A} = \omega t + \varphi$$

$$x = A \sin\ (\omega t + \varphi)$$

This is the expression of displacement of a particle executing simple harmonic motion. Here the term φ represents the initial phase or epoch, A is the amplitude of motion.

The frequency of vibration is given by

$$n = \frac{\omega}{2\pi}$$

The time period of motion is given by

$$T = \frac{2\pi}{\omega}$$

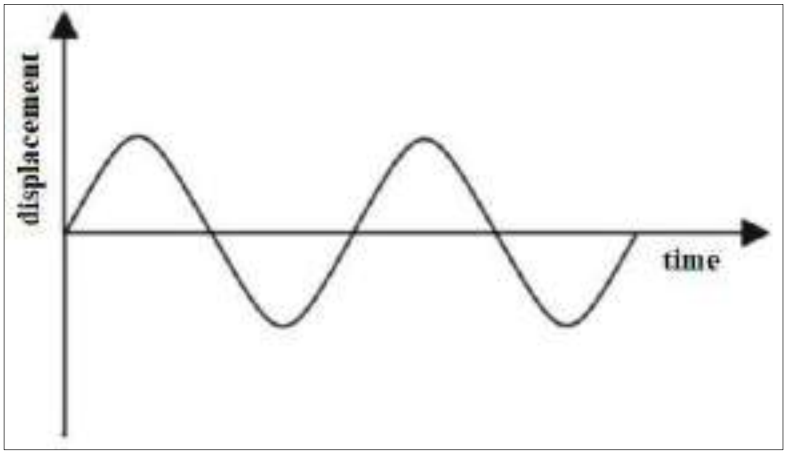


Fig. Graphical representation of displacement vs time of a particle executing SHM

4.5.3 Expression of velocity (v) of a particle executing SHM:

The expression of the displacement of a particle executing simple harmonic motion is –

$$x = A\sin\ (\omega t + \varphi)$$

Therefore, the velocity can be easily obtained by –

$$v = \frac{dx}{dt}$$

$$v = A\omega\cos\ (\omega t + \varphi)$$

Which can be written as –

$$v = \pm A\omega\sqrt{1 - sin^2(\omega t + \varphi)}$$

From the expression of displacement, we get –

$$\frac{x}{A} = \sin\ (\omega t + \varphi)$$

Using this value in the expression of velocity we get –

$$v = \pm\, A\omega\sqrt{1 - (\frac{x}{A})^2}$$

$$v = \pm\, \omega\sqrt{A^2 - x^2}$$

The maximum value of velocity: $v_{max} = \pm A\omega$ at $x = 0$

The minimum value of velocity: $v_{min} = 0$ at $x = \pm A$

The velocity becomes maximum at the equilibrium position and the velocity is minimum at the maximum of displacement.

4.5.4 Expression of acceleration (f) of a particle executing SHM:

We have just obtained the expression of velocity of a particle executing SHM –

$$v = A\omega\cos\ (\omega t + \varphi)$$

Therefore, the acceleration is given by –

$$f = \frac{dv}{dt}$$

$$f = -A\omega^2\sin(\omega t + \varphi)$$

Which can be written in terms of displacement as –

$$f = -\omega^2 x$$

The maximum value of acceleration: $f_{max} = \omega^2 A$ at $x = \pm A$

The minimum value of acceleration: $f_{min} = 0$ at $x = 0$

4.5.5 Expression of energy (E) of a particle executing SHM:

The particle executing SHM possesses both the kinetic and potential energy.

Kinetic Energy (K): The expression of kinetic energy is well known and can be written as –

$$K = \frac{1}{2}mv^2$$

The expression of velocity of a particle executing SHM is

$$v = \pm\,\omega\sqrt{A^2 - x^2}$$

$$K = \frac{1}{2}m\omega^2(A^2 - x^2)$$

The maximum value of kinetic energy: $K_{max} = \frac{1}{2}m\omega^2 A^2$ at $x = 0$

The minimum value of kinetic energy: $K_{min} = 0$ at $x = \pm A$

Potential energy (P): A restoring force is working on a particle executing SHM. Therefore we have to perform some external work to overcome the restoring force. This work will be stored as potential energy of the body.

Let the particle is at point P at any instant of time at a distance ‘x’ from the origin O.

We know the restoring force is given by –

$$F = -kx$$

The value of restoring force at P: $F = -kx$

The value of restoring force at O: $F = 0$

The average force acting on the body:

$$F_{av} = \frac{0 + kx}{2} = \frac{1}{2}kx = \frac{1}{2}m\omega^2 x$$

The work done is –

$$W = F_{av}.x = \frac{1}{2}m\omega^2 x^2$$

The expression of potential energy is –

$$P = \frac{1}{2}m\omega^2 x^2$$

The maximum value of potential energy: $P_{max} = \frac{1}{2}m\omega^2 A^2$ at $x = \pm A$

The minimum value of potential energy: $P_{min} = 0$ at $x = 0$

The expression of total energy (E):

$$E = K + P = \frac{1}{2}m\omega^2(A^2 - x^2) + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2$$

Therefore, the total energy of a body executing SHM is conserved.

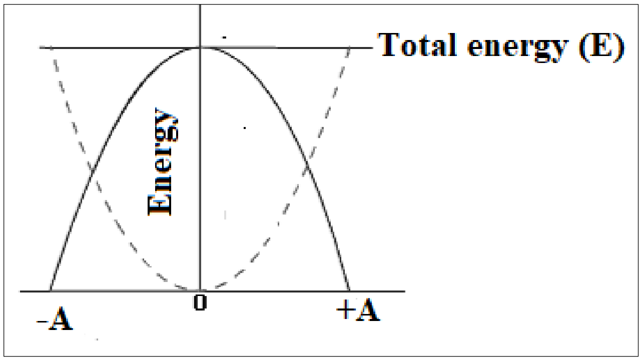


Fig: The solid curve indicates kinetic energy and the dotted line represent potential energy

The total energy of a body executing SHM is proportional to the square of the amplitude (A).

4.6. Some examples of SHM

4.6.1 Motion of a bob of simple pendulum:

Suppose a simple pendulum having a bob of mass ‘m’ is attached to a string of length ‘L’ and at any instant makes angle ‘θ’ with the vertical. At any arbitrary position following forces are acting on the bob –

i. mg – the force due to gravity acting vertically downward.

ii. T – the tension acting on the string towards the point of suspension.

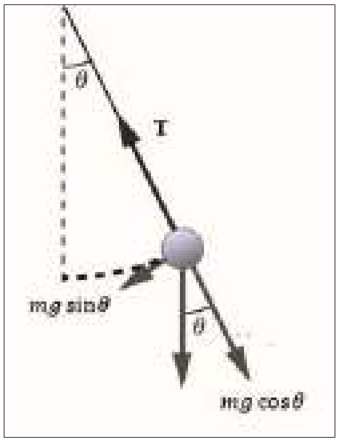


Fig: Schematic diagram of a simple pendulum

The gravitational force 'mg' has two components. One of which is 'mg cos θ' that balances the tension of the string, while the other component which acts in the perpendicular direction of the string provides the restoring force –

$$F_{restoring} = -mg\sin\theta$$

We know that, $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ..$ and which can be approximated for small value of θ.

Thus, the restoring force can be approximated as: $F_{restoring} = -mg\theta$

According to Newton's second law, the restoring force (F)= mass × acceleration

$$m\frac{d^2x}{dt^2} = -mg\theta$$

The arc length x that the bob moves through is related to the angle θ by:

$$\theta = x/L$$

$$\frac{d^2x}{dt^2} = -gx/L$$

$$\frac{d^2x}{dt^2} + \frac{g}{L}x = 0$$

The standard equation of SHM is –

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Comparing the two equation, we can say the angular frequency –

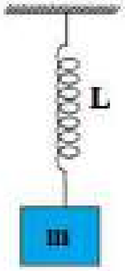
$$\omega = \sqrt{\frac{g}{L}}$$

Therefore, the motion of the bob of a simple pendulum is SHM and the time period of motion is

$$T = 2\pi \sqrt{\frac{L}{g}}$$

4.6.2. Motion of a block attached to an elastic spring:

An elastic spring having length ‘L’ is fixed in a rigid support and hanging vertically. A block of mass ‘m’ is hanging from the free end due to which the spring gets elongated.



If the block again is pulled downward and then released it starts a oscillatory motion along the vertical direction. If ‘k’ is the spring constant of the spring we can write –

$$F = -kx$$

According to Newton's second law, the restoring force (F)= mass × acceleration

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Comparing with the standard equation of SHM is – $\frac{d^2x}{dt^2} + \omega^2 x = 0$

We get – $\omega = \sqrt{\frac{k}{m}}$

Therefore the motion of the loaded spring is simple harmonic in nature and the time period is given by –

$$T = 2\pi\sqrt{\frac{m}{k}}$$

4.6.3 Motion of a liquid inside a U-shaped tube –

Consider a U-shaped tube containing a liquid is at rest such that the liquid levels on both sides are equal. Let L be the height of the liquid column from the bottom of the U-tube to the liquid surface when at equilibrium. Let A be the cross-sectional area of the tube. Let ρ be the density of the liquid. Suppose the liquid is displaced by a small amount x from the equilibrium position on one side. This will cause one side to rise by x and the other side to fall by x. The difference in height between the two sides is $2x$.

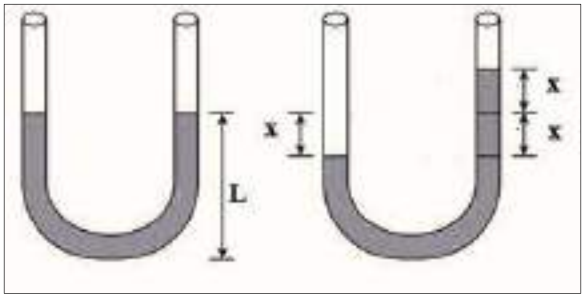


Fig: Schematic diagram of a U-shaped tube containing liquid

Hence, the force responsible for the motion of the liquid column is –

$$F = volume\ of\ the\ liquid\ of\ height\ 2x \times density\ of\ the\ liquid \times g$$

$$F = -(2xA) \times \rho \times g$$

$$F = -2xA\rho g$$

The negative sign indicates that the force is acting in the opposite direction to the rise of the liquid column and will try to depress the height.

According to Newton's second law, the restoring force (F)= mass × acceleration

Here mass is the total mass of the liquid column, which is $m = 2LA\rho$

Where 'L' is the height of liquid column in each side initially.

$$m\frac{d^2x}{dt^2} = -2xA\rho g$$

$$2LA\rho\frac{d^2x}{dt^2} = -2x\alpha\rho g$$

$$\frac{d^2x}{dt^2} + \frac{g}{L}x = 0$$

Comparing with the standard equation of SHM is –

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

We can say that the motion of the liquid column inside the U-shaped tube is SHM in nature whose angular frequency is given by -

$$\omega = \sqrt{\frac{g}{L}}$$

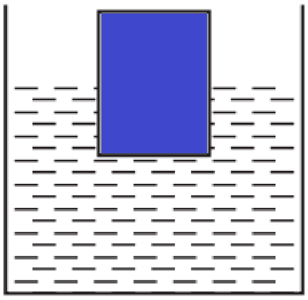
The time period can be easily written as-

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Thus, the time period of motion is independent of the density of liquid under consideration.

4.6.4 Motion of a floating block in a liquid:

Suppose a cubical block of side 'L' and density 'ρ' is floating over a liquid of density 'σ'. The block is further depressed by a length 'x' inside the liquid and then released. We have to show that the motion is simple harmonic in nature.



The volume of the displaced liquid $V = L^2x$

The mass of the displaced liquid $m = L^2x\sigma$

The buoyant force acting on the block is responsible for the motion of the block.

$$F_b = -L^2x\sigma g$$

According to Newton's second law, the restoring force (F)= mass × acceleration

Mass of the cubical block $L^3\rho$

$$m\frac{d^2x}{dt^2} = -L^2x\sigma g$$

$$L^3\rho\frac{d^2x}{dt^2} = -L^2x\sigma g$$

$$\frac{d^2x}{dt^2} + \frac{\sigma g}{\rho L}x = 0$$

Comparing with the standard equation of SHM is –

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

We can say that the motion of the cubical block is SHM in nature whose angular frequency is given by -

$$\omega = \sqrt{\frac{\sigma g}{\rho L}}$$

The time period can be easily written as-

$$T = 2\pi\sqrt{\frac{\rho L}{\sigma g}}$$

4.7 Composition of two collinear SHMs having same frequency but different amplitude and phase:

Let us suppose that the SHMs have frequency ω and amplitudes are 'a' and 'b'. The phase difference is δ. Therefore the equations of SHMS are –

$$x_1 = a\sin\omega t$$

$$x_2 = b\sin(\omega t + \delta)$$

Therefore the resultant displacement after their superposition –

$$x = x_1 + x_2$$

$$x = a\sin\omega t + b\sin(\omega t + \delta)$$

$$x = a\sin\omega t + \ b\sin\omega t\cos\delta + b\cos\omega t\sin\delta$$

Rearranging and taking common –

$$x = (a + b\cos\delta)\sin\omega t + (b\sin\delta)\ \cos\omega t$$

Let us assume – $(a + b\cos\delta) = C\cos\varphi$ and $(b\sin\delta) = C\sin\varphi$ –

$$x = C\sin(\omega t + \varphi)$$

Where, $C = \sqrt{a^2 + b^2 + 2ab\cos\delta}$ and $\tan\varphi = \frac{(b\sin\delta)}{(a+b\cos\delta)}$.

When the SHMs are in phase (δ = 0) , $C = (a + b)$ and $\varphi = 0$. The resultant motion is –

$$x = (a + b)\sin(\omega t)$$

Thus, in this case the amplitude of the resultant motion is algebraic sum of the amplitudes of the constituting SHMs.

When the SHMs are out of phase (δ = π) , $C = (a - b)$ and $\varphi = 0$. The resultant motion is –

$$x = (a - b)\sin(\omega t)$$

Thus, in this case the amplitude of the resultant motion is the difference of the amplitudes of the constituting SHMs.

4.8 Composition of two SHMs having same frequency but different amplitude and phase moving perpendicular to each other:

Let us suppose that the SHMs have frequency ω and amplitudes are ‘a’ and ‘b’. The phase difference is δ. Therefore the equations of SHMs are –

$$x = a\sin\omega t$$

$$y = b\sin(\omega t + \delta)$$

Therefore we have,

$$\frac{x}{a} = \sin\omega t$$

$$\frac{y}{b} = \sin(\omega t + \delta)$$

From the second equation –

$$\frac{y}{b} = \sin\omega t\ \cos\delta + \cos\omega t\sin\delta$$

Putting the values of $\sin\omega t$ and $\cos\omega t$ –

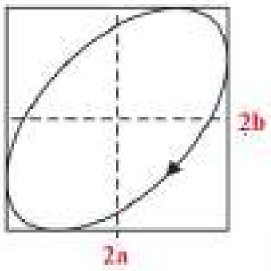
$$\frac{y}{b} = \frac{x}{a}\cos\delta \pm \sqrt{1 - \frac{x^2}{a^2}}\sin\delta$$

$$\frac{y}{b} - \frac{x}{a}\cos\delta = \pm\sqrt{1 - \frac{x^2}{a^2}}\sin\delta$$

Squaring and rearranging –

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\delta = \ sin^2\delta$$

This is a general equation of ellipse which is inclined with the cartesian x-y axis and is within the rectangular shaped box of side 2a and 2b.



Case I: For δ = 0, sin δ = 0 and cos δ = 1 which yields –

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}= 0$$

$$(\frac{x}{a}-\frac{y}{b})^2 = 0$$

$$y=\frac{b}{a}x$$

This represents a straight line passing through origin and having positive slope.

Case II: For δ = π, sin π = 0 and cos π = -1 which yields –

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2xy}{ab}= 0$$

$$(\frac{x}{a}+\frac{y}{b})^2 = 0$$

$$y=-\frac{b}{a}x$$

This represents a straight line passing through origin and having negative slope.

Case III: For δ = π/2, sin π/2 = 1 and cos π/2 = 0 which yields –

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

This represents a pure ellipse.

Case IV: For δ = π/2, sin π/2 = 1 and cos π/2 = 0 and a = b which yields –

$$x^2+y^2=a^2$$

This represents a circle.

The curves obtained by the superposition of two SHMs moving perpendicular to each other are called **Lissajous figure**. The nature of Lissajous figure depends upon the value of frequency, amplitude and phase difference of the constituent SHMs.

4.9 Damped Vibration

When a vibrating system experiences some resistive forces or frictional force, the system dissipates its energy with time and its amplitude of motion decreases with time and as a reason the motion of the body gets stopped. This type of vibration having decreasing amplitude is called damped vibration.

Example: A car suspension system where the shock absorbers provide damping to reduce oscillations after a bump.

4.9.1 Differential equation of Damped vibration:

In case of a body executing damped vibration there are two forces acting in the body –

i. Restoring force: The restoring force provided by the spring, which follows Hooke's law:

$$F_r = -ax$$

The constant a has the unit of N/m.

ii. Damping force: The resistive force provided by the damper, which is proportional to the velocity

$$F_d = -b\frac{dx}{dt}$$

The unit of b is given by N-s-m^{-1} and it is the damping force per unit velocity.

Therefore, the total force acting on the body:

$$F = F_r + F_d$$

According to Newton's law –

$$F = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - ax$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{a}{m}x = 0$$

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0$$

This is the differential equation of a body executing damped harmonic motion. We have written $\frac{b}{m} = 2k$ as it makes future calculation easier. This is a second order differential equation and to solve this we assume –

$$x = Ae^{\lambda t}$$

$$\frac{dx}{dt} = A\lambda e^{\lambda t}$$

$$\frac{d^2x}{dt^2} = A\,\lambda^2 e^{\lambda t}$$

Putting all these in the equation of damped harmonic motion we get –

$$A\,\lambda^2 e^{\lambda t} + A\lambda e^{\lambda t}2k + \omega^2 A e^{\lambda t} = 0$$

$$Ae^{\lambda t}[\lambda^2 + 2k\lambda + \omega^2] = 0$$

$$Ae^{\lambda t} \neq 0$$

$$\lambda^2 + 2k\lambda + \omega^2 = 0$$

$$\lambda = \frac{-2k \pm \sqrt{4k^2 - 4\omega^2}}{2}$$

$$\lambda = -k \pm \sqrt{k^2 - \omega^2}$$

Thus, the general solution of damped vibration is given by –

$$x = Ae^{(-k+\sqrt{k^2-\omega^2})t} + Be^{(-k-\sqrt{k^2-\omega^2})t}$$

In case of damped vibration, depending upon the damping of the system following three cases may arise –

Case I [$k > \omega$]: In this case, the damping is strong, so the system does not oscillate but returns to equilibrium. In this case the damping coefficient is higher than the natural angular frequency. In this case the general solution reduces to –

$$x = Ae^{-at} + Be^{-bt}$$

Here, a and b are real and positive. This type of motions are called overdamped motion. Heavy door with a hydraulic door closer, motion of a pendulum inside a viscous liquid are examples of overdamped motion.

Case II [$k \to \omega$]: This is the case where the damping is just enough to prevent oscillations. The system returns to equilibrium as quickly as possible without oscillating. Here, as 'k' is nearly same to 'ω' from the general solution we can write –

$$x = e^{-kt}\,(Ae^{\gamma t} + \,Be^{-\gamma t})$$

Where, $\gamma = \sqrt{k^2 - \omega^2}$ and is very small in nature.

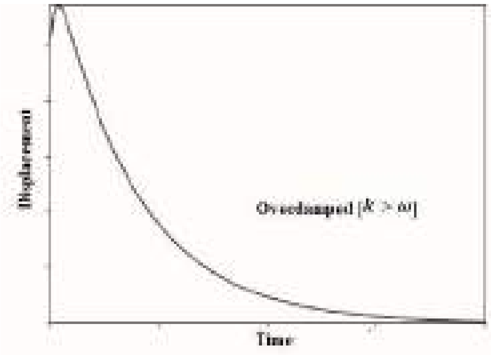
Now neglecting the higher order terms and expanding the exponential series we get –

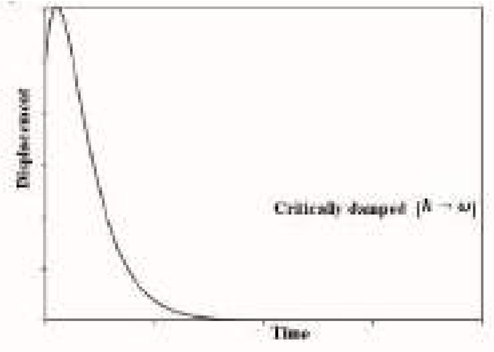
$$x = e^{-kt}[A(1 + \gamma t) + B(1 - \gamma t)]$$

$$x = e^{-kt}[(A + B) + (A - B)\gamma t]$$

$$x = e^{-kt}[C + Dt]$$

An example of critically damped motion is automotive shock absorbers in a car. In a critically damped system, the damping is just enough to return to equilibrium as quickly as possible without oscillating.





Case III [$k < \omega$]: In this case, the damping is feeble, so the system oscillates while gradually losing energy. The amplitude of oscillations decreases exponentially over time. For this case, the term $\sqrt{k^2 - \omega^2}$ becomes imaginary. So, we can rearrange the term to write –

$$\sqrt{k^2 - \omega^2} = \sqrt{-(\omega^2 - k^2)} = i\sqrt{(\omega^2 - k^2)} = i\delta$$

Where, $\sqrt{(\omega^2 - k^2)} = \delta$

$$x = e^{-kt}A(e^{(\sqrt{k^2-\omega^2})t} + Be^{(-\sqrt{k^2-\omega^2})t}$$

$$x = e^{-kt}[Ae^{i\delta t} + Be^{-i\delta t}]$$

$$x = e^{-kt}[A(\cos\delta t + i\sin\delta t) + B(\cos\delta t - i\sin\delta t)]$$

$$x = e^{-kt}[(A + B)\cos\delta t + i(A - B)\sin\delta t]$$

$$x = e^{-kt}[R\sin\varphi\cos\delta t + R\cos\varphi\sin\delta t]$$

$$x = Re^{-kt}\sin(\delta t + \varphi)$$

Here we have assumed, $A + B = R\sin\varphi$ and $i(A - B) = R\cos\varphi$

From the above expression we can see that the amplitude is proportional to e^{-kt}. The damping has the following effect to the oscillatory motion –

i. The amplitude of motion is no more constant, but decreases exponentially with time.

$$Amplitude = Re^{-kt}$$

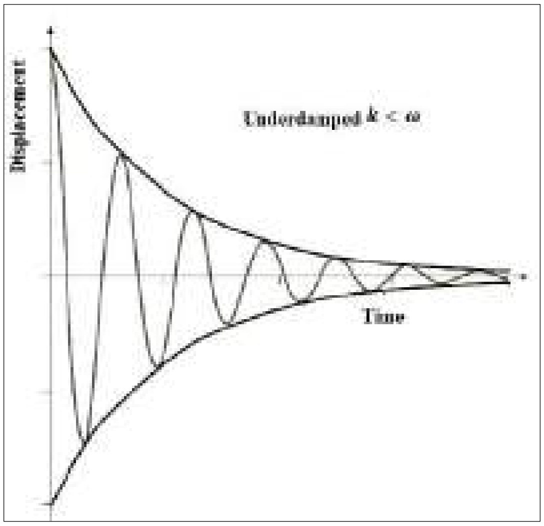
ii. The frequency of oscillation of the body decreases due to damping.

$$\delta = \sqrt{(\omega^2 - k^2)}$$

Thus, the frequency becomes –

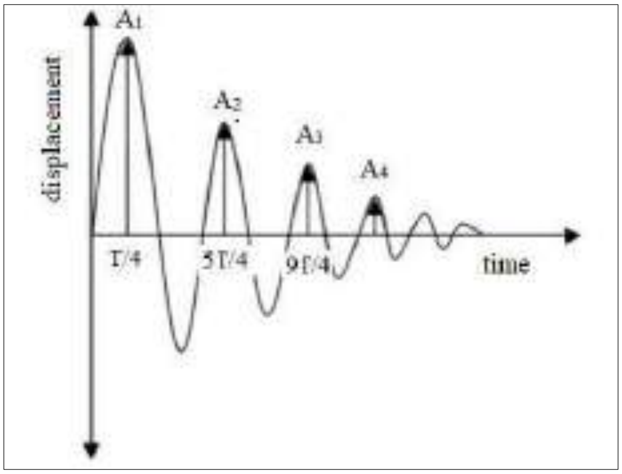
$$Frequency = \frac{\delta}{2\pi} = \frac{\sqrt{(\omega^2 - k^2)}}{2\pi} = Initial\ frequency \times \sqrt{1 - \frac{k^2}{\omega^2}}$$

The discharge of a charged capacitor through an inductive coil of low inductance is example of underdamped motion.



4.9.2 Some important parameters:

i. Logarithmic decrement: Logarithmic decrement measures the rate at which the amplitude of the oscillation of a damped harmonic oscillator decreases over time.



The displacement of a damped oscillator is given by $x = Re^{-kt}\sin(\delta t + \varphi)$

The amplitude (A) of motion is $A = Re^{-kt}$

Initially, $t = 0, A_0 = R$. We can write –

$$A = A_0\, e^{-kt}$$

$t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}, \frac{7T}{4}$ ... we have -

$$A_1 = A_0\, e^{-\frac{kT}{4}}$$

$$A_2 = A_0\, e^{-\frac{3kT}{4}}$$

$$A_3 = A_0\, e^{-\frac{5kT}{4}}$$

$$A_4 = A_0\, e^{-\frac{7kT}{4}}$$

Now we have –

$$\frac{A_1}{A_2} = \frac{A_2}{A_3} = \frac{A_3}{A_4} = \cdots = e^{\frac{kT}{2}}$$

$$d = \ e^{\frac{kT}{2}}$$

Thus the ratio of the amplitudes of any two successive peaks of the oscillating system remains constant. It is called decrement (d). The natural logarithm of decrement is called logarithmic decrement (λ).

$$\boldsymbol{\lambda = \ln d = \frac{kT}{2}}$$

ii. Relaxation time (τ)

The total energy of a particle executing damped harmonic motion is given by –

$$E = kinetic\ energy + potential\ energy$$

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2 x^2$$

We know the expression of displacement –

$$x = Re^{-kt}\sin(\delta t + \varphi)$$

$$\delta = \sqrt{(\omega^2 - k^2)}$$

$$v = \frac{dx}{dt} = Re^{-kt}\delta\cos(\delta t + \varphi) + Rke^{-kt}\sin(\delta t + \varphi)$$

Let us assume, initially - $t = 0,\ \ v = v_0\ and\ \varphi = 0$

$$v_0 = R\delta$$

$$R = v_0/\delta$$

Substituting in the expression of displacement we get –

$$x = \frac{v_0}{\delta}\ e^{-kt}\sin(\delta t + \varphi)$$

$$v = \frac{dx}{dt} = \frac{v_0}{\delta}\ e^{-kt}\delta\cos(\delta t + \varphi) + \frac{v_0}{\delta}\ e^{-kt}(-k)\sin(\delta t + \varphi)$$

For a system with very small damping $\delta \sim \omega$ and we can neglect the second term.

$$v = v_0\ e^{-kt}\cos(\omega t + \varphi)$$

$$x = \frac{v_0}{\omega}\ e^{-kt}\sin(\omega t + \varphi)$$

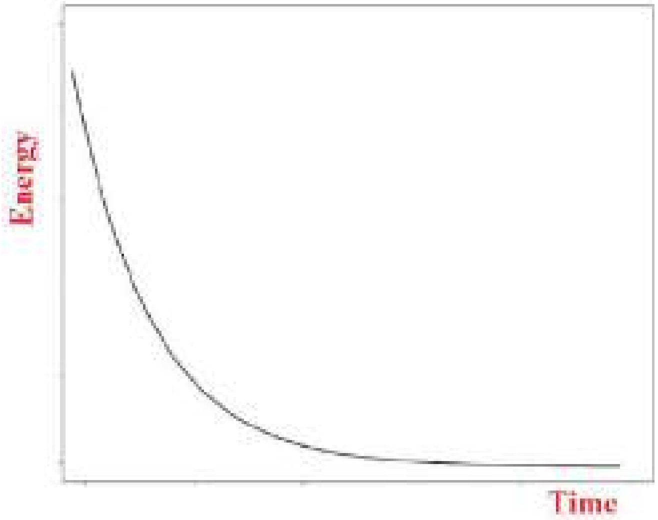
Substituting all these in the expression of total energy we get –

$$E = \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2 x^2$$

$$E = \frac{1}{2}m[v_0\ e^{-kt}\cos(\omega t + \varphi)]^2 + \frac{1}{2}m\omega^2[\frac{v_0}{\omega}\ e^{-kt}\sin(\omega t + \varphi)]^2$$

$$E = \frac{1}{2}m{v_0}^2 e^{-2kt}$$

$$E = E_0 e^{-2kt}$$



The relaxation time is a measure of how quickly a damped harmonic oscillator loses energy or, equivalently, how quickly its amplitude decreases to a certain fraction of its initial value. Specifically, the relaxation time τ is defined as the time required for the energy of the oscillation to decrease to 1/e (about 36.8%) of its initial value.

At $t = \tau, E = \frac{E_0}{e}$

$$\frac{E_0}{e} = E_0 e^{-2k\tau}$$

$$\tau = \frac{1}{2k}$$

iii. Quality Factor (Q):

Quality factor is a useful parameter to identify the damping of an oscillatory system. A higher Q factor indicates lower energy dissipation which indicates that the system oscillates for a longer time before the amplitude decays significantly. It is also defined as the ratio of the maximum value of restoring force to the maximum value of damping force. Thus, we can write –

$$Q = \frac{Maximum\ value\ of\ the\ restoring\ force}{Maximum\ value\ of\ the\ damping\ force}$$

$$Q = \frac{(F_r)_{max}}{(F_d)_{max}}$$

The expression of damping and restoring force are –

$$F_d = -b\ \frac{dx}{dt}$$

$$F_r = -ax$$

Thus the maximum value of damping force is –

$$(F_d)_{max} = -b(\frac{dx}{dt})_{max}$$

$$x = \frac{v_0}{\delta}\ e^{-kt}\sin(\delta t + \varphi)$$

$$v = \frac{dx}{dt} = v_0\ e^{-kt}\cos(\delta t + \varphi)$$

Again we know from the previous section –

$$v_0 = R\delta$$

For a system with very small damping –

$$\delta \sim \omega$$

$$v_{max} = v_0\ e^{-kt} = R\omega\ e^{-kt} = A\omega$$

$$(F_d)_{max} = -bA\omega$$

$$(F_r)_{max} = -aA$$

The values of constant ‘k’ and ‘ω’ are –

$$k = \frac{b}{2m}$$

$$\omega = \sqrt{\frac{a}{m}}$$

Substituting all the values we get

$$Q = \frac{(F_r)_{max}}{(F_d)_{max}}$$

$$Q = \frac{-aA}{-bA\omega} = \frac{m\omega^2 A}{2kmA\omega} = \frac{\omega}{2k}$$

$$Q = \frac{\omega}{2k} = \omega\tau$$

4.10. Forced vibration

When an externally applied periodic force tries to drive a system in such a way that the system starts to oscillate with the frequency of the external force rather than its natural frequency of vibration, then it is called forced vibration.

The external force is periodic in nature. The system responds in following two ways to the external periodic force –

Transient Response: This is the initial response of the system toward the external force and generally it decays over time due to damping.

Steady-State Response: This is the long-term response of the system, which occurs at the frequency of the external driving force.

The external periodic force may be taken as $F_e = F_0 e^{ipt}$

Therefore, the total force acting on the body:

$$F = F_r + F_d + F_e$$

The differential equation of forced vibration is given by –

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = f_0 e^{ipt}$$

Where, $f_0 = F_0/m$

For steady state response the differential equation may be solved for the particular integral –

Let us assume –

$$x = Ae^{ipt}$$

$$\frac{dx}{dt} = Aipe^{ipt}$$

$$\frac{d^2x}{dt^2} = A\, i^2 p^2 e^{ipt} = -A\, p^2 e^{ipt}$$

Substituting all these in the differential equation of forced vibration we get –

$$-A\, p^2 e^{ipt} + 2kAipe^{ipt} + \omega^2 Ae^{ipt} = f_0 e^{ipt}$$

$$Ae^{ipt}[-p^2 + 2kip + \omega^2] = f_0 e^{ipt}$$

$$A = \frac{f_0}{(\omega^2 - p^2) + i2kp}$$

Where,

$$(\omega^2 - p^2) = D\cos\varphi$$

$$2kp = D\sin\varphi$$

$$D = \sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}$$

$$\tan\varphi = \frac{2kp}{(\omega^2 - p^2)}$$

Thus ‘A’ can be written as –

$$A = \frac{f_0}{D\cos\varphi + iD\sin\varphi} = \frac{f_0}{D(\cos\varphi + i\sin\varphi)} = \frac{f_0}{D}e^{-i\varphi}$$

So the particular solution is –

$$x = \frac{f_0\, e^{i(pt-\varphi)}}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}}$$

For the transient response we shall solve the homogeneous differential equation –

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0$$

We already have solve this equation for damped vibration and know that only for the case $k < \omega$ that is for the underdamped case the motion is oscillatory in nature and can be written as –

$$x = Ce^{-kt}\sin(\delta t + \varphi)$$

Where

$$\delta = \sqrt{(\omega^2 - k^2)}$$

The general solution is given by –

$$x = \frac{f_0\, e^{i(pt-\varphi)}}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}} + Ce^{-kt}\sin(\delta t + \varphi)$$

As the second term contains negative exponent of time, therefore the after some time the first term will dominate and the solution for the forced vibration can be approximated to –

$$x = \frac{f_0\, e^{i(pt-\varphi)}}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}}$$

4.11. Amplitude resonance:

Amplitude resonance refers to the condition in which a system subjected to an external periodic force oscillates with maximum amplitude. At this point, the system absorbs energy most efficiently from the driving force, leading to large oscillations.

The steady state solution is given by –

$$x = \frac{f_0 \, e^{i(pt-\varphi)}}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}}$$

The expression of amplitude is –

$$A = \frac{f_0}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}}$$

The amplitude becomes maximum when the denominator is minimum. The denominator is a function of frequency of the external periodic force (p). Therefore the condition of minimum value of the denominator is –

$$\frac{df}{dp} = 0$$

Here,

$$f(p) = (\omega^2 - p^2)^2 + 4k^2p^2$$

$$\frac{df}{dp} = 2(-2p)(\omega^2 - p^2) + 2p4k^2$$

$$\frac{df}{dp} = 0$$

$$2(-2p)(\omega^2 - p^2) + 2p4k^2 = 0$$

$$p^2 = \omega^2 - 2k^2$$

$$p_r = \sqrt{\omega^2 - 2k^2}$$

At this value of the frequency of the external periodic force the amplitude becomes maximum.

$$A_{max} = \frac{f_0}{\sqrt{(\omega^2 - {p_r}^2)^2 + 4k^2{p_r}^2}}$$

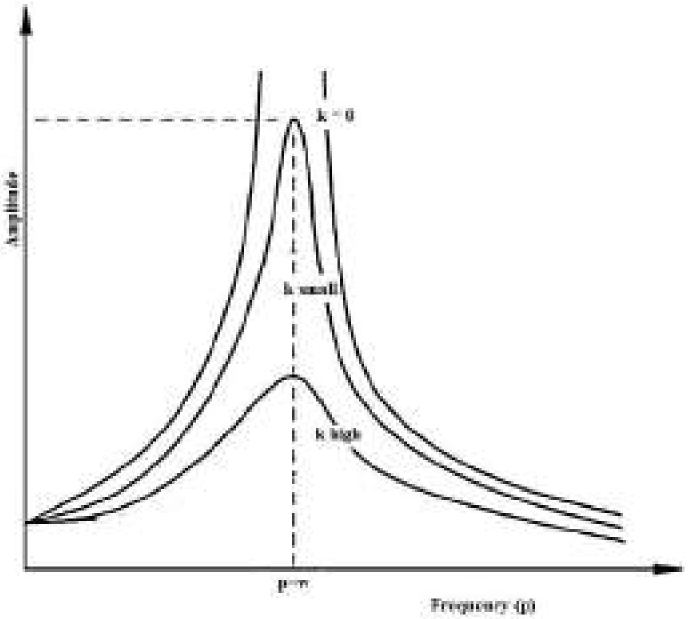
$$A_{max} = \frac{f_0}{\sqrt{(\omega^2 - (\sqrt{\omega^2 - 2k^2})^2)^2 + 4k^2(\sqrt{\omega^2 - 2k^2})^2}}$$

$$A_{max} = \frac{f_0}{\sqrt{(\omega^2 - \omega^2 + 2k^2)^2 + 4k^2(\omega^2 - 2k^2)}}$$

$$A_{max} = \frac{f_0}{\sqrt{4k^4 + 4k^2\omega^2 - 8k^4}} = \frac{f_0}{\sqrt{4k^2\omega^2 - 4k^4}} = \frac{f_0}{2k\sqrt{\omega^2 - k^2}}$$

$$A_{max} = \frac{f_0}{2k\sqrt{\omega^2 - 2k^2 + k^2}} = \frac{f_0}{2k\sqrt{{p_r}^2 + k^2}}$$

From the expression it is clear that the damping constant also play a key role in determining the sharpness of the resonance curve.



4.12. Velocity Resonance:

Velocity resonance refers to the condition in which the amplitude of the velocity of an oscillating system reaches its maximum value due to the applied periodic driving force. This typically occurs at a frequency which matches with the natural frequency of the system.

The expression of displacement for forced vibration is –

$$x = \frac{f_0 \sin(pt - \varphi)}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}}$$

Therefore the velocity is –

$$v = \frac{dx}{dt} = \frac{f_0 p \cos(pt - \varphi)}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}} = V \cos(pt - \varphi)$$

Where, ‘V’ is the velocity amplitude which is given by

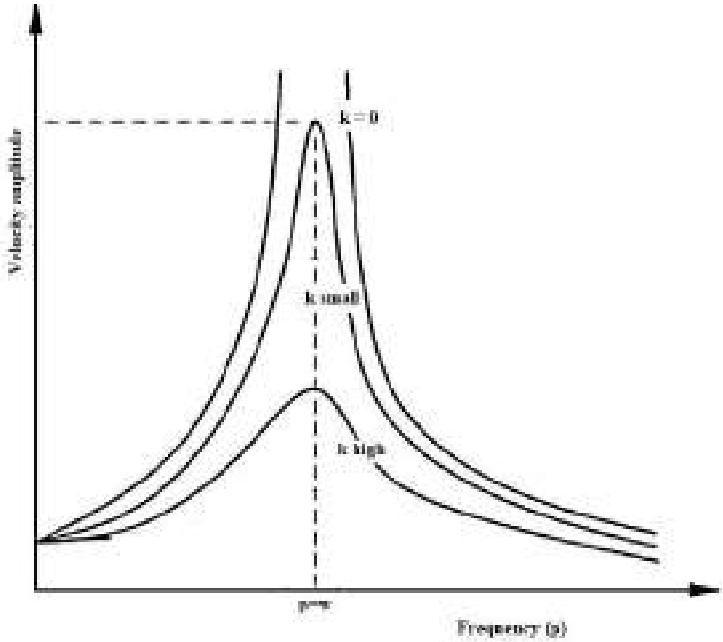
$$V = \frac{f_0 p}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}}$$

Rearranging the term we may write –

$$V = \frac{f_0}{\sqrt{\frac{(\omega^2 - p^2)^2 + 4k^2p^2}{p^2}}} = \frac{f_0}{\sqrt{(\frac{\omega^2 - p^2}{p})^2 + 4k^2}} = \frac{f_0}{\sqrt{(\frac{\omega^2}{p} - p)^2 + 4k^2}}$$

Velocity amplitude becomes maximum when the frequency of the external periodic force matches with the natural frequency of vibration $p = \pm\omega$.

$$V_{max} = \frac{f_0}{2k}$$



4.13. Sharpness of resonance:

The sharpness of resonance measures how narrowly a system responds to its resonant frequency. A higher Q factor or low value of ‘k’ indicates a sharper or narrower peak in the resonance curve, meaning the system has low damping and oscillates for a longer period once excited. Conversely, a lower Q factor or high value of ‘k’ indicates broader resonance with higher damping and quicker energy dissipation.

The sharpness of a resonance curve is measured by using the concept of Full width at Half Maxima (FWHM). In a graph, the full width at half maximum (FWHM) is the difference between the two values of the independent variable at which the dependent variable is equal to half of its maximum value. In other words, it is the width of a spectrum curve measured between those points on the y-axis which are half the maximum amplitude.

4.14. In the steady state forced vibration the rate of dissipation of energy due to resistive frictional force is equal to the rate of supply of energy by the driving force in each cycle

Let in the steady state forced vibration, the displacement of a particle is represented by –

$$x = A\sin(pt - \varphi)$$

The external periodic force is given by –

$$F = F_0 \sin pt$$

Let us suppose that the force moves the particle by amount 'dx' in time 'dt'. Therefore, the work done is -

$$dW = F_0 \sin pt \; dx$$

Therefore, the rate of work done is $F_0 \sin pt \frac{dx}{dt}$

The average rate of work done over a complete cycle is –

$$= \frac{1}{T}\int_0^T F_0 \sin pt \left(\frac{dx}{dt}\right) dt = \frac{1}{T}\int_0^T F_0 \sin pt \;\; pA\cos(pt - \varphi)\, dt$$

$$= \frac{F_0 pA}{T}\int_0^T [\sin pt \cos pt \cos\varphi + sin^2 pt \sin\varphi] dt$$

$$= \frac{1}{2} F_0 pA \sin\varphi$$

Here we have used, $T = \frac{2\pi}{p}$

Again we have,

$$x = A\sin(pt - \varphi)$$

$$v = Ap\cos(pt - \varphi) = V \;\cos(pt - \varphi)$$

V= Ap gives the peak value of the velocity or velocity amplitude

We also have –

$$\tan\varphi = \frac{2kp}{(\omega^2 - p^2)}$$

Substituting, $k = \frac{b}{2m}$ and $\omega = \sqrt{\frac{a}{m}}$ we get –

$$\tan\varphi = \frac{b}{\frac{a}{p} - mp}$$

$$\sin\varphi = \frac{b}{\sqrt{b^2 + (mp - \frac{a}{p})^2}}$$

The velocity amplitude can also be written as –

$$V = Ap = \frac{f_0 p}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}} = \frac{F_0}{\sqrt{b^2 + (mp - \frac{a}{p})^2}}$$

Putting all these back into the expression of $\frac{1}{2}F_0 pA \sin\varphi$ we get the average rate of work done over a complete cycle is –

$$\frac{1}{2}bV^2$$

Now the resistive frictional force is given by $F_d = -b\frac{dx}{dt}$

Therefore, the rate of work done against this force averaged over a complete cycle is –

$$= \frac{1}{T}\int_0^T b\frac{dx}{dt}.\left(\frac{dx}{dt}\right)dt = \frac{1}{T}\int_0^T b\,(\frac{dx}{dt})^2\,dt = \frac{1}{2}bA^2p^2 = \frac{1}{2}bV^2$$

Therefore, the rate of dissipation of energy due to resistive frictional force is equal to the rate of supply of energy by the driving force in each cycle

EXERCISE

1. Multiple Choice Type Question:

(i) The total energy of a body executing SHM is proportional to:

(a) the square of amplitude (b) square root of amplitude

(c) cube of amplitude (d) independent of amplitude.

(ii) The relation between natural angular frequency (ω) and time period (T) is:

(a) $\omega = 2\pi/T^{1/2}$ (b) $\omega = 2\pi/T$ (c) $\omega = 2\pi T$ (d) $\omega = \pi^2/T$

(iii) The general expression for kinetic energy of a body executing SHM is

(a) $E_K = \frac{1}{2}mw^2x^2$ (b) $E_K = \frac{1}{2}mw^2(a^2 - x^2)$ (c) $E_K = \frac{1}{2}mw^4x^4$ (d) $E_K = \frac{1}{2}mw^2$

(iv) The differential equation of a body of mass 'm' with damping constant 'k' and natural angular frequency 'w' is given by:

(a) $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0$ (b) $\frac{d^2x}{dt^2} + \omega^2 x = 0$

(c) $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} = 0$ (d) $\frac{d^3x}{dt^3} + m\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0$

(v) The condition for light damping or underdamping is:

(a) $k = \omega$ (b) $k > \omega$ (c) $k < \omega$ (d) $k = 0$.

(vi) The complimentary function for the differential equation for the forced vibration is given by:

(a) $x = Ce^{-kt}\sin(w't + \delta)$ (b) $x = C\sin(wt + \delta)$

(c) $x = Ce^{-kt}\,t^3\sin(w't + \delta)$ (d) none of these.

(vii) The resonance curve is sharper for:

(a) system with smaller damping factor k (b) system with higher damping factor k

(c) system with damping k tends to zero (d) none of these.

(viii) The inductance 'L' for electrical circuit is equivalent to which quantity of the acoustic circuit:

(a) mass (b) velocity (c) displacement (d) momentum

(ix) Which one of the following motion is oscillatory in nature:

(a) overdamped (b) underdamped (c) critically damped (d) none of these.

(x) The quantity half width (Δ) measures:

(a) the frequency of the external periodic force. (b) nature of the damping force.

(c) the sharpness of resonance (d) none of these.

2.Numerical Questions:

(i) Write down the differential equation of a series LCR circuit driven by a sinusoidal voltage. Identify the natural frequency of the circuit and relaxation time. Obtain the condition that this circuit will show an oscillatory decay.

(ii) A mass of 10 gm is acted upon by a restoring force 5 dyne/cm and a resistance 2 dyne cm^{-1} s. Find whether the motion is oscillatory or aperiodic. Find the value of the resistance for the motion to be critically damped.

(iii) The amplitude of an oscillator of frequency 200 cycles/s falls to 1/10th of its initial value after 2000 cycles. Calculate the relaxation time, quality factor, damping constant.

(iv) Calculate the damped frequency of oscillation and relaxation of an LCR circuit with L = 3 H, C = 0.05μF and R = 100 ohm.

(v) Consider a particle undergoing simple harmonic motion. The velocity of the particle at position x_1 is v_1 and velocity of the particle at position x_2 is v_2. Show that the ratio of time period and amplitude is –

$$\frac{T}{A} = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 x_2^2 - v_2^2 x_1^2}}$$

(vi) At an instant of time displacement of a particle is 12 cm and velocity is 5 cm/s, when its displacement is 5 cm, velocity is 12 cm/s. Calculate the amplitude, frequency and time period of the motion.

(vii) If the damping force acting on a body is of constant magnitude, show that the frequency of vibration of a damped oscillator is not affected by the magnitude of the damping.

(viii) The equation for displacement of a point of a damped oscillator is given by –

$$x = 10\, e^{-0.25t} \sin\left(\frac{\pi}{2}\right) t \ meter$$

Find the velocity of the oscillating point at t = T/4 and T, where T is the time period.

(ix) Two forced harmonic oscillations have same displacement amplitudes at the frequencies 400 rad/s and 800 rad/s. Calculate the resonant frequency at which the displacement is maximum.

(x) Show that the displacement vs velocity graph of a simple harmonic oscillator is elliptical.

❖ ❖ ❖ ❖ ❖ ❖ **THE END** ❖ ❖ ❖ ❖ ❖ ❖